Differential geometry with SageMath

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Outline

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Introduction

- Computer algebra system (CAS) started to be developed in the 1960's; for instance `Macsyma` (to become `Maxima` in 1998) was initiated in 1968 at MIT
- In 1965, J.G. Fletcher developed the `GEOM` program, to compute the Riemann tensor of a given metric
- In 1969, during his PhD under Pirani supervision, Ray d'Inverno wrote `ALAM (Atlas Lisp Algebraic Manipulator)` and used it to compute the Riemann tensor of Bondi metric. The original calculations took Bondi and his collaborators 6 months to go. The computation with ALAM took 4 minutes and yielded to the discovery of 6 errors in the original paper [J.E.F. Skea, *Applications of SHEEP* (1994)]
- Since then, many software tools for tensor calculus have been developed... A rather exhaustive list: `http://www.xact.es/links.html`

Outline

SageMath in a few words

- SageMath (*nickname:* Sage) is a **free open-source** mathematics software system
- it is based on the Python programming language
- it makes use of **many pre-existing open-sources packages**, among which
  - Maxima, Pynac (symbolic calculations)
  - GAP (group theory)
  - PARI/GP (number theory)
  - Singular (polynomial computations)
  - matplotlib (high quality 2D figures)

  and provides a **uniform interface** to them
- William Stein (Univ. of Washington) created SageMath in 2005; since then, $\sim$**100 developers** (mostly mathematicians) have joined the SageMath team
- SageMath is now supported by European Union via the open-math project OpenDreamKit (2015-2019, within the *Horizon 2020* program)

The mission

Create a viable free open source alternative to Magma, Maple, Mathematica and Matlab.

Some advantages of SageMath

SageMath is free

Freedom means

1. everybody can use it, by downloading the software from http://sagemath.org
2. everybody can examine the source code and improve it

SageMath is based on Python

- no need to learn any specific syntax to use it
- easy access for students
- Python is a very powerful *object oriented language*, with a neat syntax

SageMath is developing and spreading fast

...sustained by an enthusiastic community of developers

Object-oriented notation in Python

As an object-oriented language, Python (and hence SageMath) makes use of the following **postfix notation** (same in C++, Java, etc.):

```
result = object.function(arguments)
```

In a procedural language, this would be written as

```
result = function(object,arguments)
```

Examples

1. `riem = g.riemann()`
2. `lie_t_v = t.lie_der(v)`

NB: no argument in example 1

Outline

The SageManifolds project

http://sagemanifolds.obspm.fr/

Aim

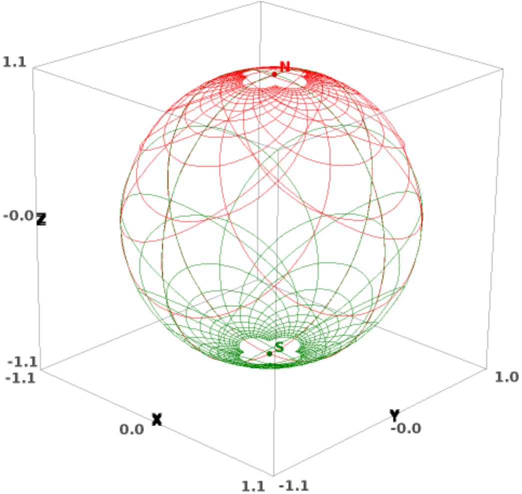
Implement smooth manifolds of arbitrary dimension in SageMath and tensor calculus on them

In particular:

- one should be able to introduce an arbitrary number of coordinate charts on a given manifold, with the relevant transition maps
- tensor fields must be manipulated as such and not through their components with respect to a specific (possibly coordinate) vector frame

Concretely, the project amounts to creating new Python classes, such as `TopologicalManifold`, `DifferentiableManifold`, `Chart`, `TensorField` or `Metric`, within SageMath's **Parent/Element framework**.

The 2-sphere example



Stereographic coordinates on the 2-sphere

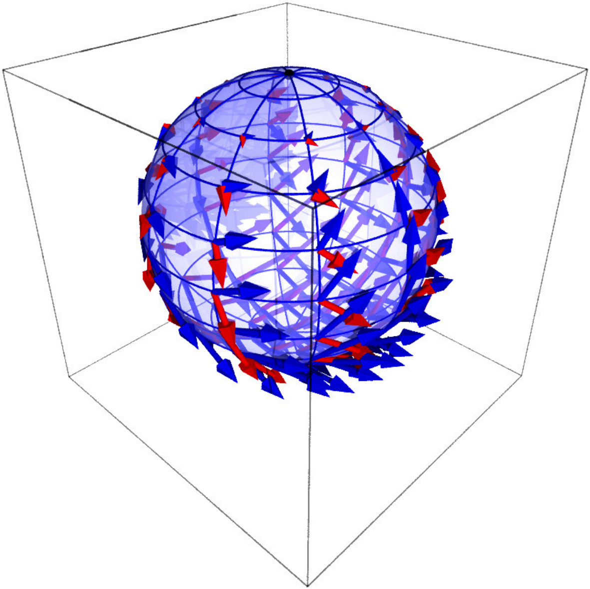
Two charts:

- X_1: $\mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$
- X_2: $\mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$

$\leftarrow$ picture obtained via function `RealChart.plot()`

See the worksheet at http://sagemanifolds.obspm.fr/examples.html

The 2-sphere example



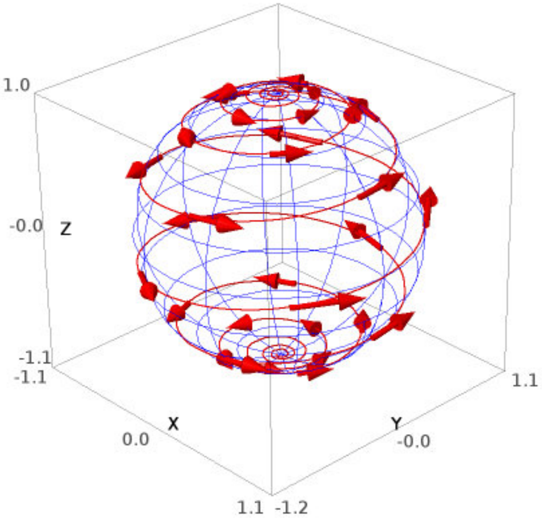
Vector frame associated with the stereographic coordinates (x, y) from the North pole

- $\frac{\partial}{\partial x}$
- $\frac{\partial}{\partial y}$

← picture obtained via the function `VectorField.plot()`

See the worksheet at http://sagemanifolds.obspm.fr/examples.html

The 2-sphere example

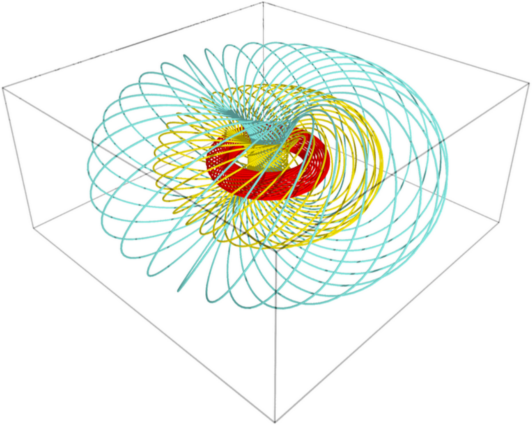


A curve in $\mathbb{S}^2$: a loxodrome and its tangent vector field

← picture obtained via the functions `DifferentiableCurve.plot()` and `VectorField.plot()`

See the worksheet at http://sagemanifolds.obspm.fr/examples.html

The 3-sphere example

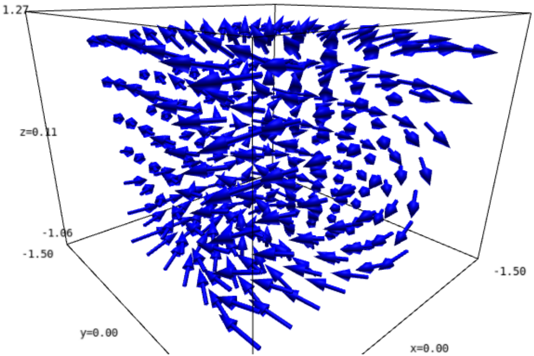


Some fibers of the **Hopf fibration** of $\mathbb{S}^3$ viewed in stereographic coordinates

← picture obtained via the function `DifferentiableCurve.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_Hopf.ipynb

The 3-sphere example



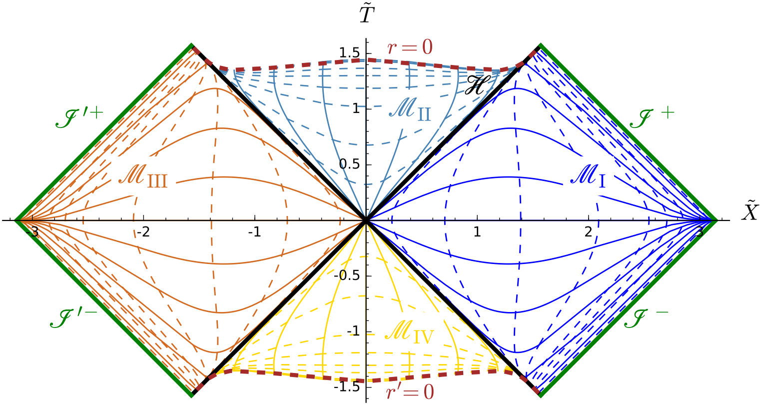
One of the vector fields of a left-invariant global vector frame of $\mathbb{S}^3$, viewed in stereographic coordinates

← picture obtained via the function `VectorField.plot()`

See the worksheet at http://nbviewer.jupyter.org/github/sagemanifolds/SageManifolds/blob/master/Worksheets/v1.0/SM_sphere_S3_vectors.ipynb

Charts on Schwarzschild spacetime

The Carter-Penrose diagram



Two charts of standard Schwarzschild-Droste coordinates (t, r, θ, φ) plotted in terms of Frolov-Novikov compactified coordinates $(\tilde{T}, \tilde{X}, \theta, \varphi)$; see the worksheet at http://luth.obspm.fr/~luthier/gourgoulhon/bh16/sage.html

Outline

Summary

SageManifolds: extends the modern computer algebra system SageMath towards differential geometry and tensor calculus

- http://sagemanifolds.obspm.fr/
- free software (GPL), as SageMath
- $\sim$ 65,000 lines of Python code (including comments and doctests)
- submitted to SageMath community as a sequence of 14 tickets
  $\rightarrow$ first ticket accepted in March 2015,
  the 14th one in Nov. 2016
- 5 developers, 3 reviewers

SageManifolds 1.0 released on 11 Jan. 2017 and fully included in SageMath 7.5

SageManifolds 1.0.1 released on 25 March 2017 and fully incl. in SageMath 7.6

Current status

Already present (v1.0):

- topological manifolds: charts, open subsets, maps between manifolds, scalar fields
- differentiable manifolds: tangent spaces, vector frames, tensor fields, curves, pullback and pushforward operators
- standard tensor calculus (tensor product, contraction, symmetrization, etc.), even on non-parallelizable manifolds
- taking into account any monoterm tensor symmetry
- exterior calculus (wedge product, exterior derivative, Hodge duality)
- Lie derivatives of tensor fields
- affine connections (curvature, torsion)
- pseudo-Riemannian metrics
- some plotting capabilities (charts, points, curves, vector fields)
- parallelization (on tensor components) of CPU demanding computations, via the Python library `multiprocessing`

Current status

Future prospects:

- extrinsic geometry of pseudo-Riemannian submanifolds
- computation of geodesics (numerical integration via SageMath/GSL or `Gyoto`)
- integrals on submanifolds
- more graphical outputs
- more functionalities: symplectic forms, fibre bundles, spinors, variational calculus, etc.
- connection with numerical relativity: using SageMath to explore numerically-generated spacetimes

Want to join the project or simply to stay tuned?

visit http://sagemanifolds.obspm.fr/
(download, documentation, example worksheets, mailing list)