Averaging of disturbed Friedmann–Lemaitre cosmological model

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Outline

- The model and its properties
- Averaging with weak limit
- Construction of the background

Motivation

- Being inspired by the perturbation theory, the aim is to construct a disturbed version of the Friedmann–Lemaitre cosmological model by simple generalization of its metric.
- We are interested in a model with the following properties
  - metric is lorentzian
  - energy density is positive
  - growing inhomogeneities are allowed
  - inhomogeneities are finite
- What can and what can not be achieved by performing such a construction?
- Is a model, even unphysical, with such properties possible? If it exists, what are its limitations?
- Explicit models can be used for testing different averaging strategies.

- The metric, the matter four-velocity and the Einstein equations are

$$g_{\mu\nu} = \begin{pmatrix} -b^2 e^{ce} & 0 & 0 & 0 \\ 0 & a^2 e^{cd(x)} & 0 & 0 \\ 0 & 0 & a^2 e^{cd(y)} & 0 \\ 0 & 0 & 0 & a^2 e^{cd(z)} \end{pmatrix}, \qquad u_\nu = \begin{pmatrix} -be^{\frac{1}{2}ce} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\kappa T_{\mu\nu} = G_{\mu\nu},$$

where

$$a = t^{\varkappa}, \qquad b = \varpi t^{\frac{3}{2}\varkappa\gamma - 1}, \qquad c = \varepsilon t^{\delta}, \qquad d(z) = \sin^{2\beta}\left(\frac{z}{\lambda}\right) + \alpha,$$

$$e = \frac{3}{2}(\varsigma_1 + \alpha)F, \qquad F = f + \frac{2}{3}\frac{\varsigma_2 - \varsigma_1^2}{(\varsigma_1+\alpha)^2}, \qquad \varsigma_s = \frac{(2s\beta)!}{2^{2s\beta}((s\beta)!)^2}.$$

- Spacetime of the model is lorentzian, inhomogeneous and does not satisfy the weak energy condition.

- Energy density in the model is non-negative, infinite in zero and finite in infinity when

$$\delta < 0 \text{ and}$$
$$\Big(\varkappa < 0 \wedge \gamma \leqslant 0 \wedge F \leqslant 0 \wedge (\varepsilon < 0 \wedge \alpha \leqslant -1 \vee 0 < \varepsilon \wedge 0 \leqslant \alpha) \text{ or}$$
$$0 < \varkappa \wedge 0 \leqslant \gamma \wedge 0 \leqslant F \wedge (\varepsilon < 0 \wedge 0 \leqslant \alpha \vee 0 < \varepsilon \wedge \alpha \leqslant -1)\Big) \text{ or}$$
$$0 < \delta \text{ and}$$
$$\Big(\varkappa < 0 \wedge \gamma < 0 \wedge (F < 0 \vee F = 0 \wedge 2\delta \leqslant 3\varkappa\gamma) \wedge (\varepsilon < 0 \wedge 0 \leqslant \alpha \vee 0 < \varepsilon \wedge \alpha \leqslant -1) \text{ or}$$
$$0 < \varkappa \wedge 0 < \gamma \wedge (0 < F \vee F = 0 \wedge 2\delta \leqslant 3\varkappa\gamma) \wedge (\varepsilon < 0 \wedge \alpha \leqslant -1 \vee 0 < \varepsilon \wedge 0 \leqslant \alpha)\Big).$$

- Quantities, ${}^3R_{\mu\nu}$, $H_{\mu\nu}$, η_ν, $\omega_{\mu\nu}$, q_ν vanish. The anisotropic stress tensor is given by

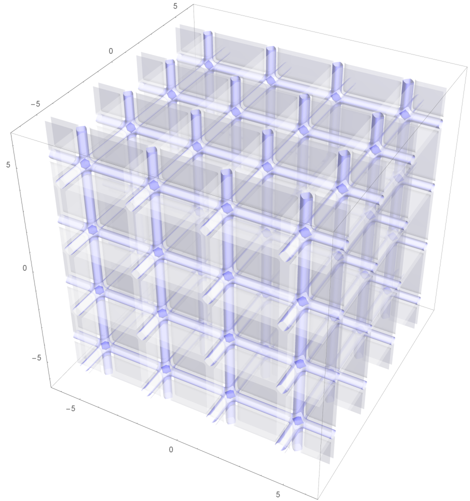
$$\kappa\pi_{\mu\nu} = u^\alpha \nabla_\alpha \sigma_{\mu\nu} + \theta\sigma_{\mu\nu},$$

and the energy density

$$\kappa\epsilon = \tfrac{1}{3}\theta^2 - \tfrac{1}{2}\sigma^{\beta\alpha}\sigma_{\beta\alpha}.$$

The model

- Isodensity surfaces



The model

- There are nine parameters in the model, α, β, γ, δ, ε, f, $\varkappa$, ϖ, λ.
- Parameters ε, ϖ are irrelevant due to scaling symmetry in the model. Physical quantities (e.g. age) are invariant under the transformation

$$\delta \to \nu\mu^{-\frac{\gamma}{\frac{2}{3}\frac{\delta}{\varkappa}}}\delta, \qquad \varepsilon \to \mu\varepsilon, \qquad \varkappa \to \nu\mu^{-\frac{\gamma}{\frac{2}{3}\frac{\delta}{\varkappa}}}\varkappa, \qquad \varpi \to \nu\varpi.$$

- The parameter λ scales spatial coordinates. The distance between overdensities is $\lambda\pi$.
- The parameter δ controls the growth rate of inhomogeneities. They grow when it is positive.

- The inhomogeneities are convex when α is positive. This parameter controls also the density contrast, $\frac{\epsilon|_{x,y,z=\lambda\pi/2}}{\epsilon|_{x,y,z=0}} - 1$. It grows from 0 at zero time to $\frac{1}{\alpha}(2+\frac{1}{\alpha})$ at infinity and is finite.
- The parameter β controls the width of the inhomogeneities. Theirs width at half maximum decreases from $\frac{2}{\pi}\arccos(\frac{1}{2})^{\frac{1}{2\beta}}$ at zero to $\frac{2}{\pi}\arccos(\sqrt{\frac{1}{2}+\alpha+\alpha^2}-\alpha)^{\frac{1}{2\beta}}$ at infinity.
- The matter in the model evolves in time. There is a limited control over its behavior. The adiabatic index of the cosmic fluid, $\frac{p}{\epsilon}+1$, changes from γ to f. The quotient $\frac{\delta}{\varkappa}$ decides on the character of the phase transition (smooth or violent).
- The shear viscosity coefficient, the ratio of $\pi^{\mu}{}_{\nu}$ to $-2\theta\sigma^{\mu}{}_{\nu}$, changes from $\frac{1}{4}(\gamma-\frac{2}{3}\frac{\delta}{\varkappa}-2)$ to $\frac{1}{4}(f+\frac{2}{3}\frac{\varsigma_2-\varsigma_1^2}{(\varsigma_1+\alpha)^2}-2)$. For reasonable values of parameters, it is negative at zero time.
- It is possible to match the basic property of the Universe that at the Hubble constant near 70 km/Mpc/s its age is about 13 Gyr.

- We average the model by averaging individual fields in it. We use the weak limit as a tool (Burnett (1989), Green and Wald (2011))

$$\lim_{\lambda\to 0}\int f^{a_1\cdots a_n}\Big(A_{a_1\ldots a_n}(\lambda)-\operatorname*{w-lim}_{\lambda\to 0}A_{a_1\ldots a_n}(\lambda)\Big)=0.$$

For example, $\text{w-lim}_{\lambda\to 0}\sin^2(\frac{z}{\lambda})=\frac{1}{2}$.

- Here, it is needed the following

$$\operatorname*{w-lim}_{\lambda\to 0}\sin^{2s\beta}\left(\frac{z}{\lambda}\right)\exp\left(t\sin^{2\beta}\left(\frac{z}{\lambda}\right)\right)$$
$$=\varsigma_s F\left(\left\{\frac{2n-1}{2\beta}+s:n=1,2,\ldots,\beta\right\},\left\{\frac{n}{\beta}+s:n=1,2,\ldots,\beta\right\},t\right).$$

- The metric does not converge uniformly as $\lambda \to 0$. We need the weak limit to average it. Due to this, the Green–Wald theorems do not apply here.
- For averaged scalar quantities there occurs that

$$\kappa\langle\epsilon\rangle = \frac{1}{3}\langle\theta^2\rangle - \frac{1}{2}\langle\sigma^{\beta\alpha}\sigma_{\beta\alpha}\rangle = \frac{1}{3}\langle\theta\rangle^2.$$

- The averaging of tensor quantities reveals that

$$\langle\sigma_{\mu\nu}\rangle \neq 0, \qquad \text{but} \qquad \langle\sigma^{\mu}{}_{\nu}\rangle = 0, \qquad \langle\pi_{\mu\nu}\rangle \neq 0, \qquad \text{but} \qquad \langle\pi^{\mu}{}_{\nu}\rangle = 0.$$

  The weak limit depends on the valence of the averaged tensor.
- According to this, we should question that $\langle g_{\mu\nu}\rangle$ can be taken as the proper background metric.

The background model

- The Green–Wald approach

$$\langle G_{\mu\nu}(g)\rangle = G_{\mu\nu}(\bar{g}) - \kappa t_{\mu\nu}.$$

  Taking $\langle g_{\mu\nu}\rangle$ as the background metric is unjustified. When $\bar{g}$ and $t_{\mu\nu}$ are free this equation could be satisfied by infinitely many ways.

- For our model, there is one distinguished solution which could serve as the background model. It is given by the metric which reproduces the averaged mixed energy-momentum tensor alone

$$\langle G^{\mu}{}_{\nu}(g)\rangle = G^{\mu}{}_{\nu}(\bar{g}).$$

  We assume here that the weak limit acting on a tensor with mixed indices gives reliable results.

The background model

- It is equivalent to searching for a metric reproducing averaged energy density $\langle \epsilon \rangle$ and pressure $\langle p \rangle$.
- We assume that the background metric has the Robertson–Walker symmetry

$$\bar{g}_{\mu\nu} = \begin{pmatrix} -B^2 & 0 & 0 & 0 \\ 0 & A^2 & 0 & 0 \\ 0 & 0 & A^2 & 0 \\ 0 & 0 & 0 & A^2 \end{pmatrix}.$$

- The equations are as follows

$$\frac{d}{dt} \ln A = -\frac{1}{3} \frac{\frac{d}{dt}\langle \epsilon \rangle}{\langle \epsilon \rangle + \langle p \rangle}, \qquad B = -\frac{2}{3} \frac{\frac{d}{dt}\sqrt{\langle \epsilon \rangle}}{\langle \epsilon \rangle + \langle p \rangle}.$$

  Right hand side of the differential equation is a rational function of time so it can be integrated explicitly.
- It appears that the expansion scalar is also reproduced, $\langle \theta(g) \rangle = \theta(\bar{g})$.

Outlook

- What are optical properties of the background model in comparison to the original model?
- What criterion should be applied to select the background model properly?
- How such an unphysical model could resemble so well the true Universe?