Averaging general inhomogeneous fluids on arbitrary spatial foliations in Relativistic Cosmology

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Inhomogeneous Cosmology Workshop
Toruń
4th July 2017

Introduction: Averaging and foliations

I – Geometrical setting and fluid content

II – Averaging the 3+1 Einstein scalar equations

III – Toward a more fluid-intrinsic approach

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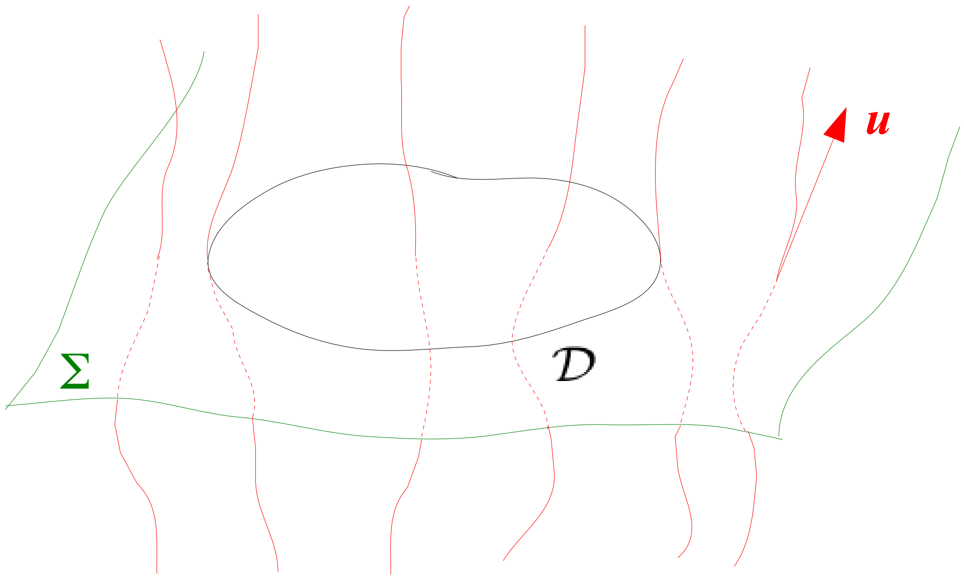


$$\mathcal{V}_{\mathcal{D}} \equiv \int_{\mathcal{D}} \sqrt{h(t,x^i)}\, \mathrm{d}^3 x$$

$$\langle \psi \rangle_{\mathcal{D}} \equiv \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \psi(t,x^i) \sqrt{h(t,x^i)}\, \mathrm{d}^3 x$$

Fluid-orthogonal foliation (irrotational flow)

$\downarrow$

Arbitrary spatial foliations ?
Generalize the system of averaged scalar Einstein equations
→ allows for other choices of spatial sections and a more general fluid content

Buchert, T.: On average properties of inhomogeneous fluids in general relativity: dust cosmologies, Gen. Rel. Grav. **32**, 105 (2000)

Buchert, T.: On average properties of inhomogeneous fluids in general relativity: perfect fluid cosmologies, Gen. Rel. Grav. **33**, 1381 (2001)

Several proposals have already been suggested:

Kasai, M., Asada, H., Futamase, T.: Toward a no-go theorem for an accelerating universe through a nonlinear backreaction, Progr. Theor. Phys. **115**, 827 (2006); Tanaka, H., Futamase, T.: A phantom does not result from a backreaction, Progr. Theor. Phys. **117**, 183 (2007)

Larena, J.: Spatially averaged cosmology in an arbitrary coordinate system, Phys. Rev. D **79**, 084006 (2009)

Brown, I.A., Behrend, J., Malik, K.A.: Gauges and cosmological backreaction, J. Cosmol. Astropart. Phys., JCAP0911:027 (2009)

Gasperini, M., Marozzi, G., Veneziano, G.: Gauge invariant averages for the cosmological backreaction, J. Cosmol. Astropart. Phys., JCAP0903:011 (2009); Gasperini, M., Marozzi, G., Veneziano, G.: A covariant and gauge invariant formulation of the cosmological "backreaction", J. Cosmol. Astropart. Phys., JCAP1002:009 (2010)

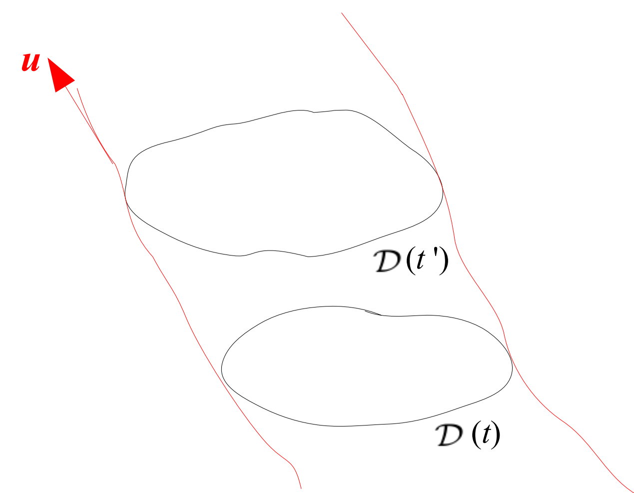
Räsänen, S.: Light propagation in statistically homogeneous and isotropic universes with general matter content, J. Cosmol. Astropart. Phys., JCAP1003:018 (2010)

Beltrán Jiménez, J., de la Cruz-Dombriz, Á., Dunsby, P.K.S., Sáez-Gómez, D.: Backreaction mechanism in multifluid and extended cosmologies, J. Cosmol. Astropart. Phys., JCAP1405:031 (2014)

Smirnov, J.: Gauge-invariant average of Einstein equations for finite volumes, *[arXiv:1410.6480]* (2014)

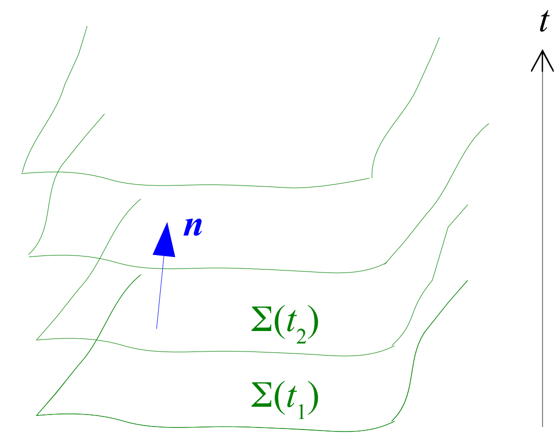
For regional domains, **domain propagation** is crucial to the dynamics.

It should be **Lagrangian**.



Geometrical setting and fluid content

Lapse and Shift



Globally hyperbolic structure

Choice of a foliation ↔ Choice of the normal vector field $\boldsymbol{n}$, irrotational (Frobenius theorem)

$(n^\mu n_\mu = -1)$

Adapted coordinates set (t, x^i):
time is constant on each hypersurface and labels them: $\Sigma(t)$; arbitrary spatial coordinates x^i

→ in these coordinates:

$$\boxed{n^\mu = \frac{1}{N}\left(1, -N^i\right) \ , \quad n_\mu = -N\,(1,0)}$$

lapse (set by foliation choice and time normalization)

shift (can be set through the propagation of the spatial coordinates)

Conventions:
- $c = 1$
- metric signature $(-,+,+,+)$
- Greek letters for space-time indices (0 to 3), Latin letters for spatial indices (1 to 3)

Fluid content — Tilt

Universe filled with a **single fluid**, characterized by its velocity field $\boldsymbol{u}$, rest-mass density ρ and general energy-momentum tensor

$$T_{\mu\nu} = \epsilon\, u_\mu u_\nu + 2\, q_{(\mu} u_{\nu)} + p\, b_{\mu\nu} + \pi_{\mu\nu}$$

energy density

heat vector

isotropic pressure

(traceless) anisotropic pressure

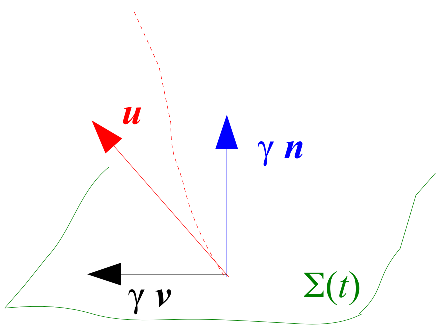
$$(b_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu)$$

Decomposition of $\boldsymbol{u}$ with respect to the foliation:

$$\boldsymbol{u} = \gamma\,(\boldsymbol{n} + \boldsymbol{v})$$

with $n^\mu v_\mu = 0$ (**tilt vector**)

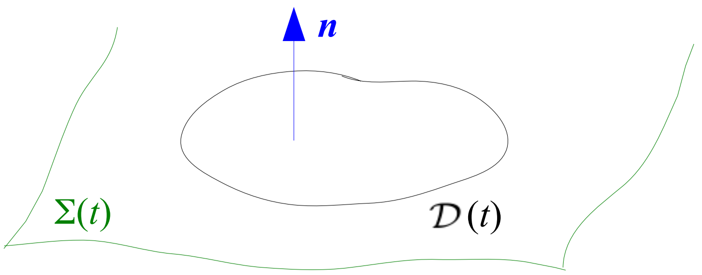
and $\gamma = -n^\mu u_\mu = \dfrac{1}{\sqrt{1 - v^\mu v_\mu}}$ (**tilt factor** or Lorentz factor)



Averaging the 3+1 Einstein scalar equations

$$\mathcal{V}_{\mathcal{D}} \equiv \int_{\mathcal{D}} \sqrt{h(t,x^i)}\,\mathrm{d}^3 x$$

$$\langle \psi \rangle_{\mathcal{D}} \equiv \frac{1}{\mathcal{V}_{\mathcal{D}}} \int_{\mathcal{D}} \psi(t,x^i)\,\sqrt{h(t,x^i)}\,\mathrm{d}^3 x$$



$$a_{\mathcal{D}}(t) \equiv \left(\frac{\mathcal{V}_{\mathcal{D}}(t)}{\mathcal{V}_{\mathcal{D}_{\mathrm{i}}}} \right)^{1/3}$$

\+ fluid-comoving domain

$$\longrightarrow \quad \frac{1}{a_{\mathcal{D}}} \frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t} = \frac{1}{3} \Big\langle -N\mathcal{K} + \left(N v^i\right)_{||i} \Big\rangle_{\mathcal{D}}$$

trace of the extrinsic curvature

spatial covariant derivative

$\longrightarrow$ **commutation rule**:

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi \rangle_{\mathcal{D}} = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t} \psi \Big\rangle_{\mathcal{D}} - \Big\langle -N\mathcal{K} + \left(N v^i\right)_{||i} \Big\rangle_{\mathcal{D}} \langle \psi \rangle_{\mathcal{D}} + \Big\langle \Big(-N\mathcal{K} + \left(N v^i\right)_{||i} \Big)\, \psi \Big\rangle_{\mathcal{D}}$$

3+1 Einstein equations (scalar part):

$$\partial_t\big|_{x^i}\mathcal{K} = N\left[\mathcal{R} + \mathcal{K}^2 - 4\pi G\,(3E - S) - 3\Lambda\right] - N^{||i}{}_{||i} + N^i\,\mathcal{K}_{||i}$$

$$\mathcal{R} + \mathcal{K}^2 - \mathcal{K}^i{}_j\mathcal{K}^j{}_i = 16\pi G\,E + 2\Lambda$$

hypersurfaces intrinsic curvature scalar — extrinsic curvature — Eulerian energy density — Eulerian pressure (x3)

→ averaging:

$$\boxed{\begin{aligned} 3\,\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}^2 a_{\mathcal{D}}}{\mathrm{d}t^2} &= -4\pi G\left\langle N^2\,(\epsilon + 3p)\right\rangle_{\mathcal{D}} + \left\langle N^2\right\rangle_{\mathcal{D}}\Lambda + \mathcal{Q}_{\mathcal{D}} + \mathcal{P}_{\mathcal{D}} + \frac{1}{2}\mathcal{T}_{\mathcal{D}}\,; \\ 3\left(\frac{1}{a_{\mathcal{D}}}\frac{\mathrm{d}a_{\mathcal{D}}}{\mathrm{d}t}\right)^2 &= 8\pi G\left\langle N^2\epsilon\right\rangle_{\mathcal{D}} + \left\langle N^2\right\rangle_{\mathcal{D}}\Lambda - \frac{1}{2}\left\langle N^2\,\mathcal{R}\right\rangle_{\mathcal{D}} - \frac{1}{2}\mathcal{Q}_{\mathcal{D}} - \frac{1}{2}\mathcal{T}_{\mathcal{D}} \end{aligned}}$$

Backreaction terms appear:

kinematical: $\mathcal{Q}_{\mathcal{D}} \equiv \left\langle N^2\left(\mathcal{K}^2 - \mathcal{K}_{ij}\mathcal{K}^{ij}\right)\right\rangle_{\mathcal{D}} - \frac{2}{3}\left\langle -N\mathcal{K} + \left(Nv^i\right)_{||i}\right\rangle_{\mathcal{D}}^2$

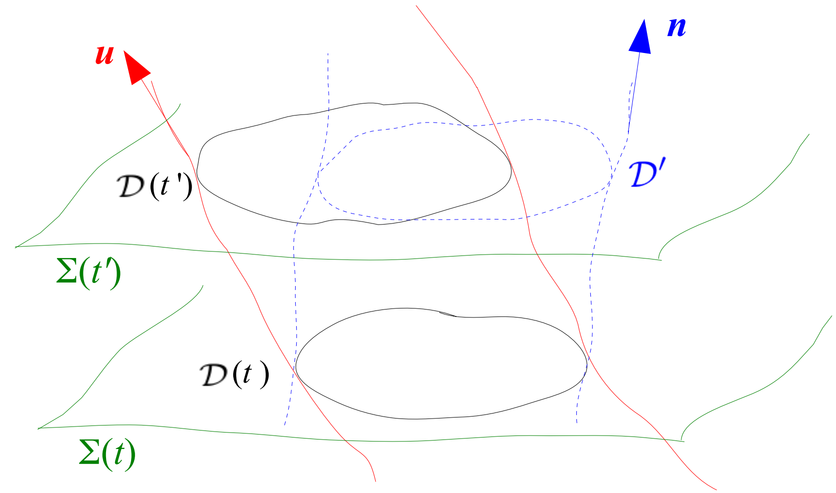
dynamical: $\mathcal{P}_{\mathcal{D}} \equiv \left\langle N N^{||i}{}_{||i} - \mathcal{K}\,\frac{\mathrm{d}N}{\mathrm{d}t}\right\rangle_{\mathcal{D}} + \left\langle \left(\left(Nv^i\right)_{||i}\right)^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\left(Nv^i\right)_{||i}\right) - 2N\mathcal{K}\left(Nv^i\right)_{||i} - N^2 v^i\mathcal{K}_{||i}\right\rangle_{\mathcal{D}}$

tilt: $\mathcal{T}_{\mathcal{D}} \equiv -16\pi G\left\langle N^2\left((\gamma^2 - 1)(\epsilon + p) + 2\,\gamma v^\alpha q_\alpha + v^\alpha v^\beta \pi_{\alpha\beta}\right)\right\rangle_{\mathcal{D}}$

Assuming usual energy conditions and negligible heat vector contribution, $\mathcal{T}_{\mathcal{D}} \leq 0$.

+ integrability condition and averaged energy conservation equation

Recalls published proposals,
but here **domain propagation is Lagrangian**.



Domain propagation matters
for a regional domain $\mathcal{D} \neq \Sigma$:
different system considered!

Available proposals: regional domains
propagate along $\boldsymbol{n}$ (or ∂_t) :

1) foliation- or coordinate-dependent considered space-time tube

2) no preservation over time of particle content / rest-mass

No such issues for Lagrangian domains (propagation along $\boldsymbol{u}$);
only solution for fluid content preservation

→ some formal differences in the equations, important difference in interpretation

More intrinsic to the fluid.

Toward a more fluid-intrinsic approach

The previous equations featured geometric, Eulerian quantities
(*e.g.* extrinsic and intrinsic hypersurface curvatures)

→ one may use fluid rest-frame kinematic quantities instead:

$$\nabla_\mu u_\nu = \underbrace{-u_\mu\, a_\nu}_{\text{acceleration}} + \frac{1}{3}\underbrace{\Theta}_{\text{expansion scalar}} b_{\mu\nu} + \underbrace{\sigma_{\mu\nu}}_{\text{shear}} + \underbrace{\omega_{\mu\nu}}_{\text{vorticity}}$$

E.g. for the commutation rule:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi\rangle_{\mathcal{D}} = \left\langle\frac{\mathrm{d}}{\mathrm{d}t}\psi\right\rangle_{\mathcal{D}} - \left\langle\frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\rangle_{\mathcal{D}}\langle\psi\rangle_{\mathcal{D}} + \left\langle\left(\frac{N}{\gamma}\Theta - \frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)\psi\right\rangle_{\mathcal{D}}}$$

$$\Rightarrow \boxed{\begin{aligned} 3\left(\frac{1}{a_\mathcal{D}}\frac{\mathrm{d}a_\mathcal{D}}{\mathrm{d}t}\right)^2 &= 8\pi G\left\langle \frac{N^2}{\gamma^2}\epsilon\right\rangle_\mathcal{D} + \Lambda\left\langle\frac{N^2}{\gamma^2}\right\rangle_\mathcal{D} - \frac{1}{2}\left\langle\frac{N^2}{\gamma^2}\mathscr{R}\right\rangle_\mathcal{D} - \frac{1}{2}\mathcal{Q}^\mathrm{T}_\mathcal{D} \\ \frac{3}{a_\mathcal{D}}\frac{\mathrm{d}^2 a_\mathcal{D}}{\mathrm{d}t^2} &= -4\pi G\left\langle\frac{N^2}{\gamma^2}(\epsilon+3p)\right\rangle_\mathcal{D} + \Lambda\left\langle\frac{N^2}{\gamma^2}\right\rangle_\mathcal{D} + \mathcal{Q}^\mathrm{T}_\mathcal{D} + \mathcal{P}^\mathrm{T}_\mathcal{D} \end{aligned}}$$

with new kinematic and dynamical backreactions:

$$\mathcal{Q}^\mathrm{T}_\mathcal{D} \equiv \frac{2}{3}\left[\left\langle\left(\frac{N}{\gamma}\Theta\right)^2\right\rangle_\mathcal{D} - \left\langle\frac{N}{\gamma}\Theta\right\rangle_\mathcal{D}^2\right] - 2\left\langle\frac{N^2}{\gamma^2}\sigma^2\right\rangle_\mathcal{D} + 2\left\langle\frac{N^2}{\gamma^2}\omega^2\right\rangle_\mathcal{D} + \frac{2}{3}\left(2\left\langle\frac{N}{\gamma}\Theta\right\rangle_\mathcal{D}\left\langle\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\rangle_\mathcal{D} - \left\langle\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right\rangle_\mathcal{D}^2\right)$$

$$\mathcal{P}^\mathrm{T}_\mathcal{D} \equiv \left\langle\frac{N^2}{\gamma^2}\nabla_\mu a^\mu\right\rangle_\mathcal{D} + \left\langle\Theta\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{N}{\gamma}\right)\right\rangle_\mathcal{D} + \left\langle 2\left(\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\right)^2 - 2\frac{N}{\gamma}\Theta\times\frac{1}{\gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t} - \frac{1}{\gamma}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\right\rangle_\mathcal{D}$$

Clearer dependence in the physical variables of the fluid...

but still several contributions from the tilt (→ foliation dependence)

→ Modifying the formalism to reduce this dependence in the tilting of the hypersurfaces ?

… To be continued!

Summary

→ importance of a Lagrangian averaging domain

→ a system of equations expressing the average evolution of a Lagrangian domain valid for a general fluid, and in any foliation

→ several possible formulations with different focusses (more or less intrinsic to the fluid) and levels of explicit dependence in the foliation behaviour

→ the quantitative results will still depend on the (free) foliation choice in any case:

in concrete applications, a particular choice of hypersurfaces has to be made, based on physical relevance:

– fluid-orthogonal (for an irrotational fluid) ?

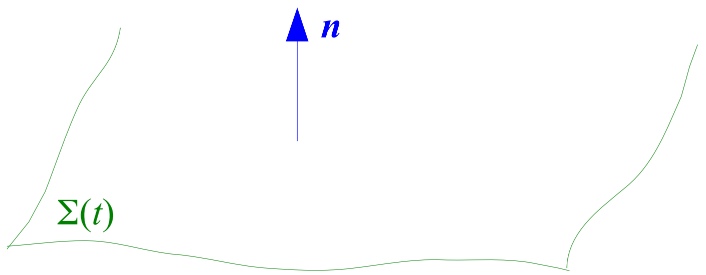
– constant rest-mass/energy density ?

– constant curvature ?

– statistical homogeneity ?

– synchronized fluid elements ?

– ...



THANK YOU FOR YOUR ATTENTION!