Backreaction for perfect fluid spheres

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Plan

Motivation

- Seek for backreaction in non–cosmological context
- Build a simple perfect fluid solution in spherical symmetry
- Study deviations from an equation of state

Method

Steps of construction of the solution:

- Postulate density profile as a perturbation of the known solution
- Solve Einstein equations (usually not possible analytically)
- Compare new solution with background

Remarks:

- These are not models of relativistic stars (static + unstable)
- Background solution in not *homogeneous*
- Backreaction = deviations from background EOS

Equations

Metric form:

$$\mathrm{d}s^2 = -\zeta^2(r)\mathrm{d}t^2 + \frac{1}{B(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2\,, \tag{1}$$

perfect fluid source:

$$T_{\mu\nu} = u_\mu u_\nu(\rho + p) + p g_{\mu\nu}\,, \tag{2}$$

Einstein equations:

$$rB'(r)\left[r\zeta'(r) + \zeta(r)\right] - 2B(r)\left\{r\left[\zeta'(r) - r\zeta''(r)\right] + \zeta(r)\right\} + 2\zeta(r) = 0\,, \tag{3}$$

$$\rho(r) = \frac{1 - B(r) - rB'(r)}{8\pi r^2}\,, \tag{4}$$

$$p(r) = \frac{\left[B(r) - 1\right]\zeta(r) + 2rB(r)\zeta'(r)}{8\pi\zeta(r)r^2}\,. \tag{5}$$

Background solutions

Schwarzschild Interior:

$$\rho(r) = \rho_0 = const \tag{6}$$

Tolman VII:

$$\rho(r) = \rho_0 \left[1 - \mu \left(\frac{r}{r_b} \right)^2 \right] \tag{7}$$

Steps:

- Solve equations for $B(r)$, $\zeta(r)$, $p(r)$
- Use junction conditions with Schwarzschild at some $r = r_b$

Perturbed Tolman VII

Perturbed density:

$$\rho(r) = \rho_{background} + A\cos(\lambda r) \tag{8}$$

Solution to 1st order ODE:

$$B(r) = 1 + \frac{C}{r} + 8\pi\rho_c\left[\frac{2A\sin(\lambda r)}{\lambda^3 r} - \frac{2A\cos(\lambda r)}{\lambda^2} - \frac{Ar\sin(\lambda r)}{\lambda} + \frac{\mu r^4}{5r_b^2} - \frac{1}{3}r^2\right] \tag{9}$$

2nd order ODE to solve:

$$\left(rB'(r) - 2B(r) + 2\right)\zeta(r) + r\left(rB'(r) - 2B(r)\right)\zeta'(r) + 2rB(r)\zeta''(r) = 0 \tag{10}$$

- Second order ODE with junction conditions at $r_{b,inh}$
- Associate coordinate radii of the perturbed solution and the background by physical distance: time of flight of the photon
- Problem with free boundary — solved iteratively

Result: backreaction depends on mass difference.

$$\delta m = m_{inh} - m_h = 4\pi \rho_c \left[\frac{A r_b^2 \sin(\lambda r_b)}{\lambda} - \frac{2A \sin(\lambda r_b)}{\lambda^3} + \frac{2A r_b \cos(\lambda r_b)}{\lambda^2} \right] . \quad (11)$$

Take $\lambda = \frac{2\pi n + \phi_0}{r_b}$:

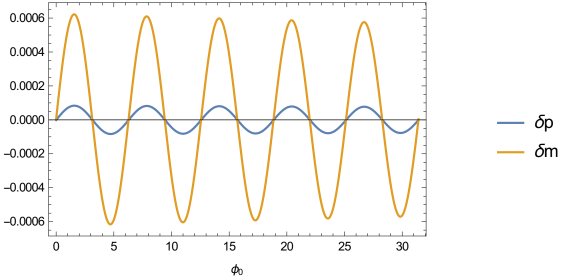
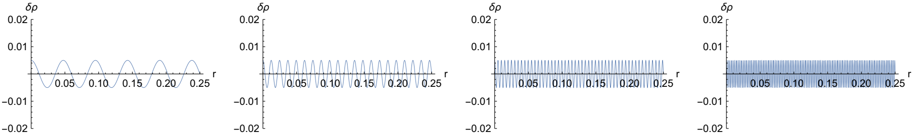
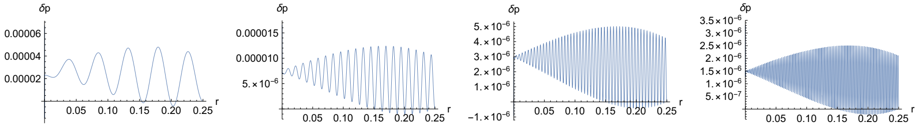


Figure: Impact of phase factor ϕ_0 on the central pressure and mass differences ($A = \frac{1}{50}$, $n = 50$).

Maximal backreaction



(A) Density difference ($A = \frac{1}{100}$, $\phi = \frac{\pi}{2}$, n=5, 20, 50, 100)



(B) Pressure difference ($A = \frac{1}{100}$, $\phi = \frac{\pi}{2}$, n=5, 20, 50, 100)

Figure: Backreaction for various frequencies of inhomogeneities

Equation of state and energy conditions

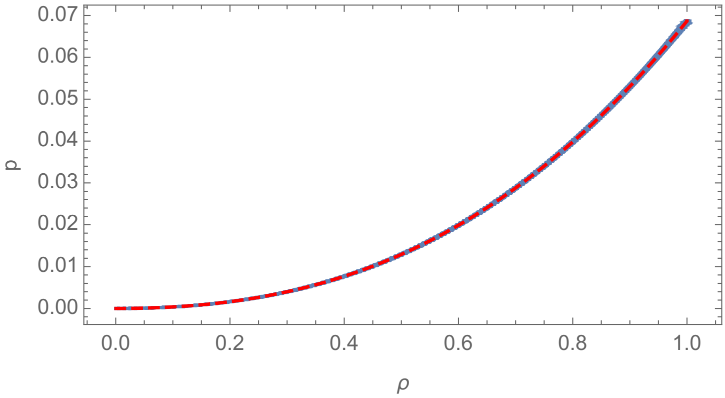


Figure: Blue: density–pressure relation for perturbed Tolman VII ($A = \frac{1}{100}$, $n = 100$). Red: density–pressure relation for exact Tolman VII.

Energy conditions: all fulfilled.

Green–Wald scheme

Assumptions:

- We are in the a family of coordinates $x^\mu(\lambda)$ such that the metric $g_{\mu\nu}(\lambda)$ takes form (1)
- $g_{\mu\nu}(\lambda)$ fulfills Einstein equations for $\lambda > 0$ and is uniformly convergent to some $g^{(0)}_{\alpha\beta}$ as $\lambda \to 0$

We want to compare:

$$\rho_0(r) = \frac{1 - B_0(r) - rB_0'(r)}{8\pi r^2}\,, \tag{12}$$

$$p_0(r) = \frac{[B_0(r) - 1]\zeta_0(r) + 2rB_0(r)\zeta_0'(r)}{8\pi\zeta_0(r) r^2}\,, \tag{13}$$

with

$$\bar{\rho}(r) = \operatorname*{w-lim}_{\lambda\to 0} \frac{1 - B_\lambda(r) - rB_\lambda'(r)}{8\pi r^2}\,, \tag{14}$$

$$\bar{p}(r) = \operatorname*{w-lim}_{\lambda\to 0} \frac{[B_\lambda(r) - 1]\zeta_\lambda(r) + 2rB_\lambda(r)\zeta_\lambda'(r)}{8\pi\zeta_\lambda(r) r^2}\,. \tag{15}$$

Observation

Let $\lim_{\lambda\to 0} g(\lambda, r) = g_0(r)$ and $\lim_{\lambda\to 0} h(\lambda, r) = h_0(r)$.[a] Then $\underset{\lambda\to 0}{\text{w-lim}}[g(\lambda, r)h'(\lambda, r)] = g_0(r)h_0'(r)$.

[a]Derivatives of these functions are not necessarily uniformly convergent.

We adapt this observation to $\bar{\rho}(r)$ and $\bar{p}(r)$ and conclude that:

$$\bar{\rho}(r) = \underset{\lambda\to 0}{\text{w-lim}}\, \rho_\lambda(r) = \rho_0(r)\,, \tag{16}$$

$$\bar{p}(r) = \underset{\lambda\to 0}{\text{w-lim}}\, p_\lambda(r) = p_0(r)\,, \tag{17}$$

This can be extended to coordinate frames, to which transformation from $x^\mu(\lambda)$ is non–singular in the limit $\lambda \to 0$.

Conclusions

1. Although the backreaction was appearing, it's order was very small (Maximally $\frac{\delta p}{p} \sim \frac{1}{1000}$ in the center of the star).
2. Backreaction (devitation from the density–pressure relation) may have different forms within this model
3. Backreaction is connected to the specific choice of the "background"
4. Green–Wald framework is giving trivial backreaction in static, spherically symmetric case in a certain class of coordinate frames.

Thank you