



# Cosmology

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  - standard model: **density perturbations (anisotropy)**
  - scalar (GR) averaging: statistically homogeneous spatial slices

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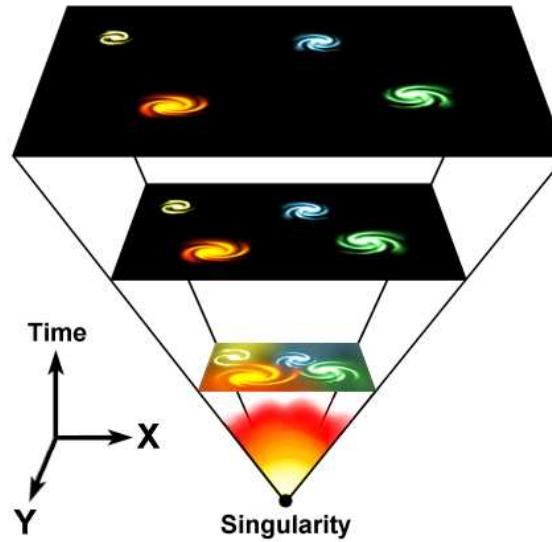
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3. assume that  $(M, \mathbf{g})$  remains unchanged if we add density perturbations to an early time slice



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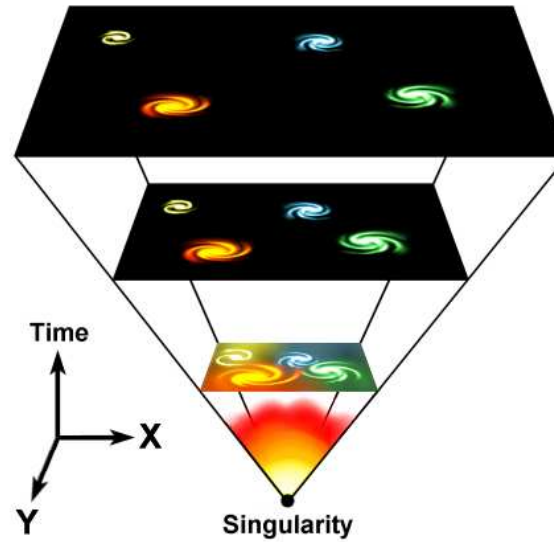
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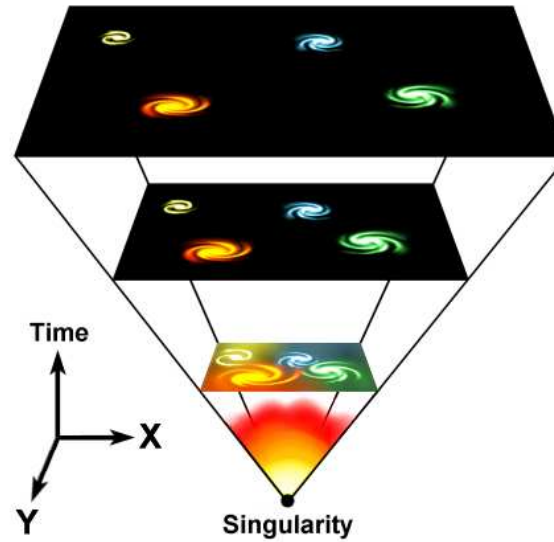


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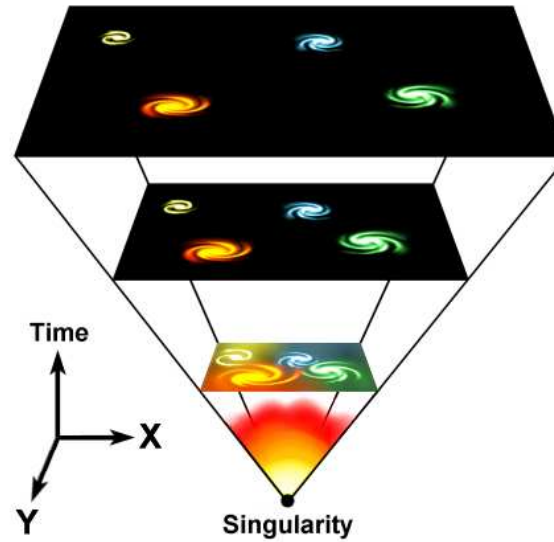
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- spherical coordinates for spatial slice

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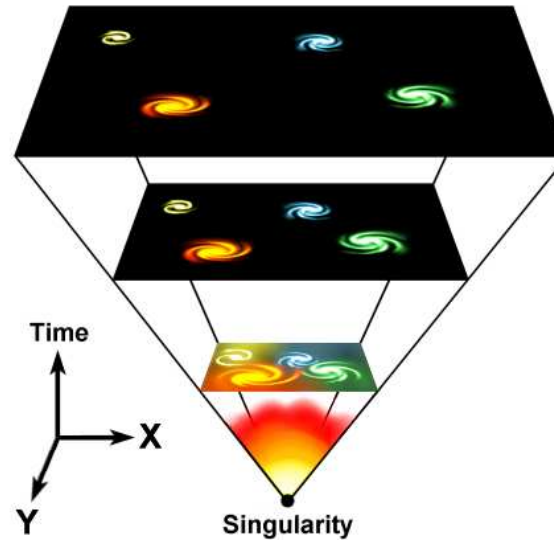


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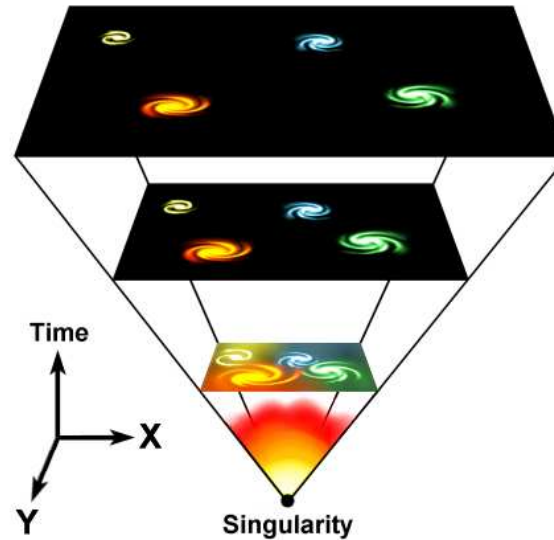
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- universe is static in comoving coordinates  $(r, \theta, \phi)$

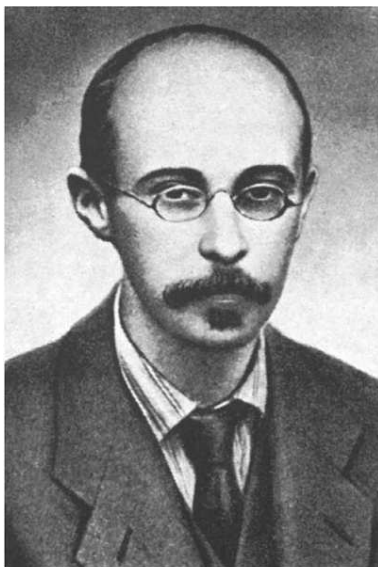
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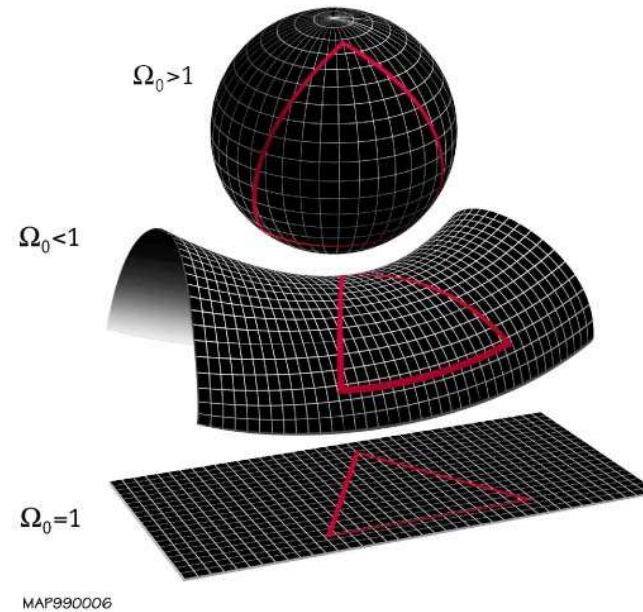
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- but  $\int du \neq$  proper time; *more*: [arXiv:astro-ph/0707.2106](https://arxiv.org/abs/0707.2106)

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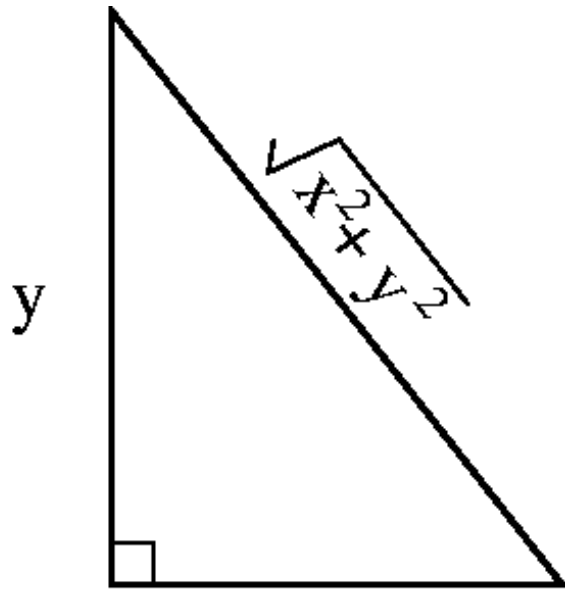
where  $r_{\perp} :=$

$$\begin{cases} R_C \sinh \frac{r}{R_C} & k < 0 \\ r & k = 0 \\ R_C \sin \frac{r}{R_C} & k > 0 \end{cases}$$

for a comoving radius of curvature  $R_C$  and curvature of sign  $k$

# curvature

- on a spatial slice (fixed value of  $t$ ):

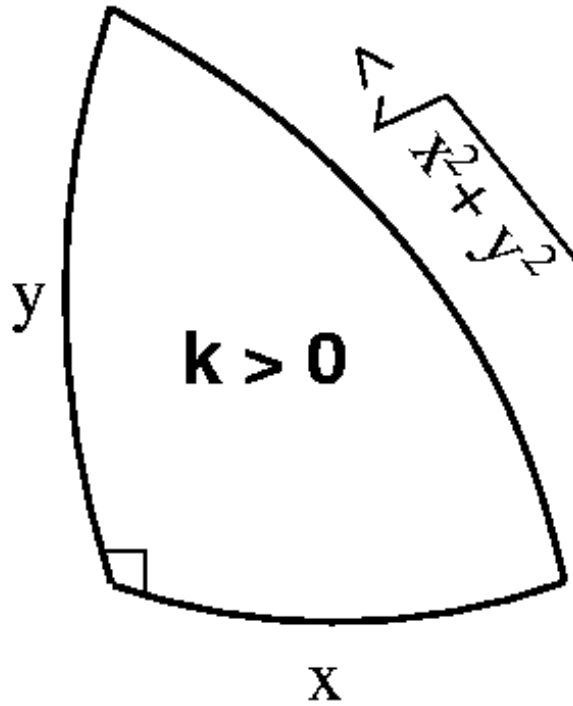


$x$

$$k = 0$$

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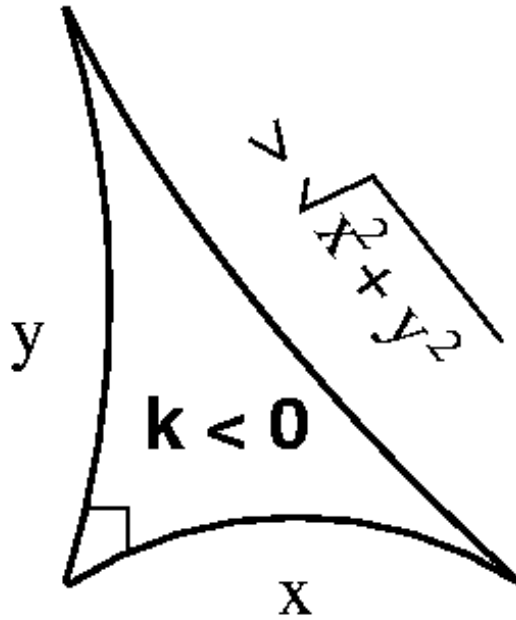
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# 2D curvature intuition: $k > 0$

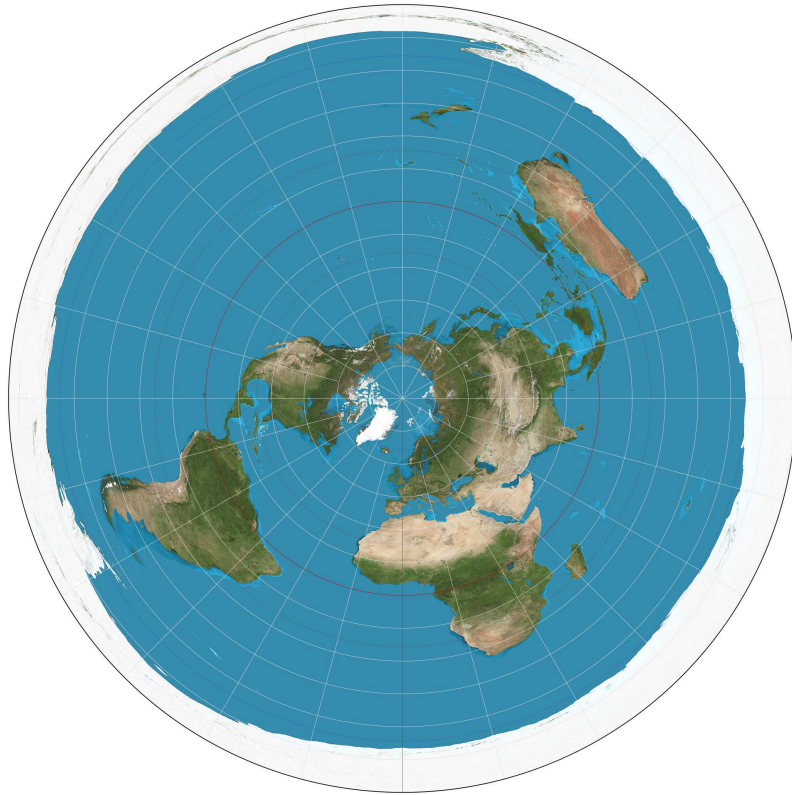
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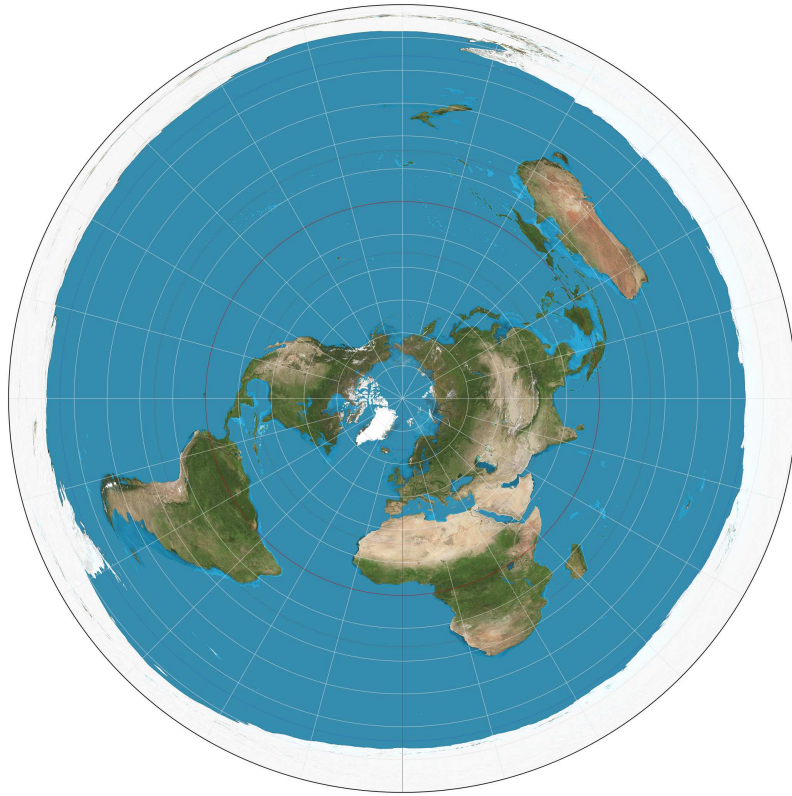
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- intuition switch:  $S^2$  easier vs  $S^3$  more physical

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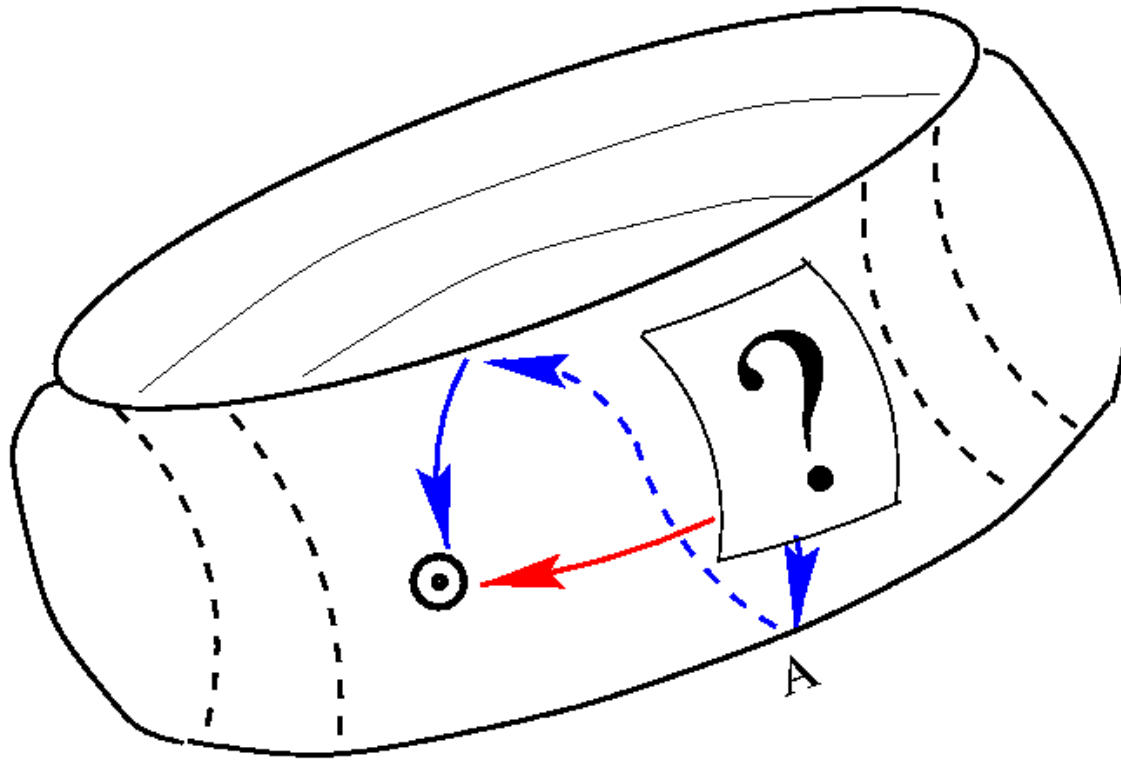
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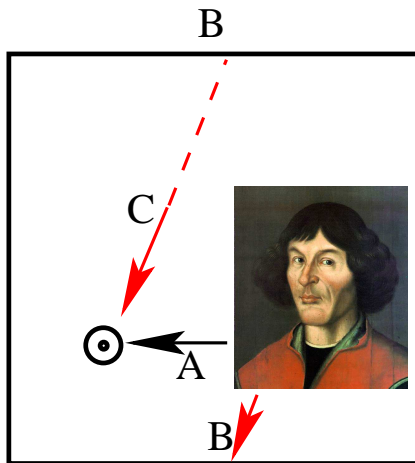


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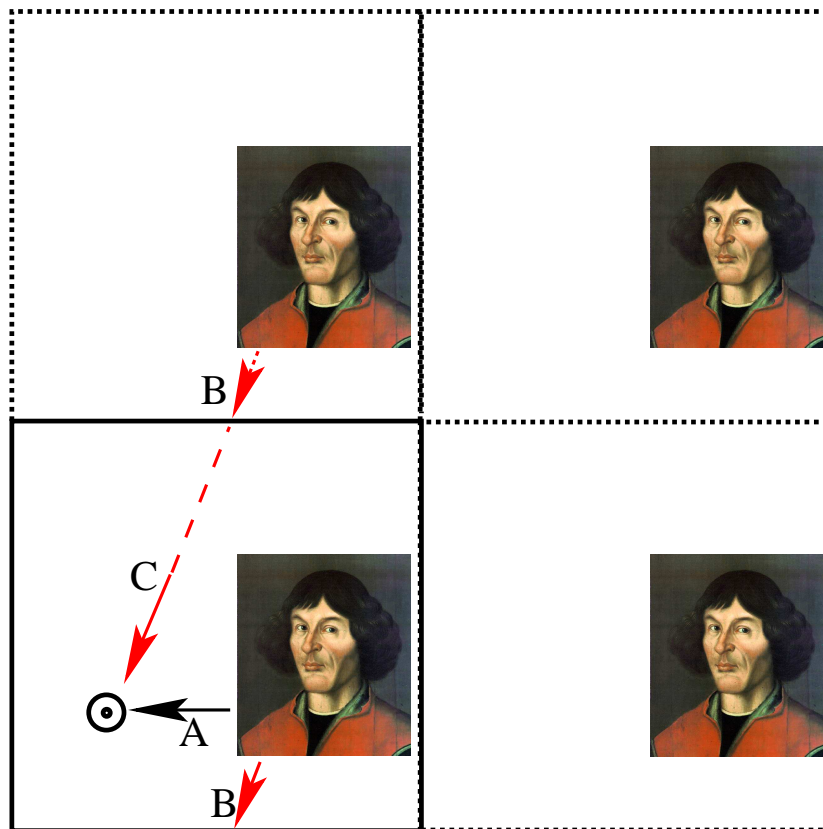


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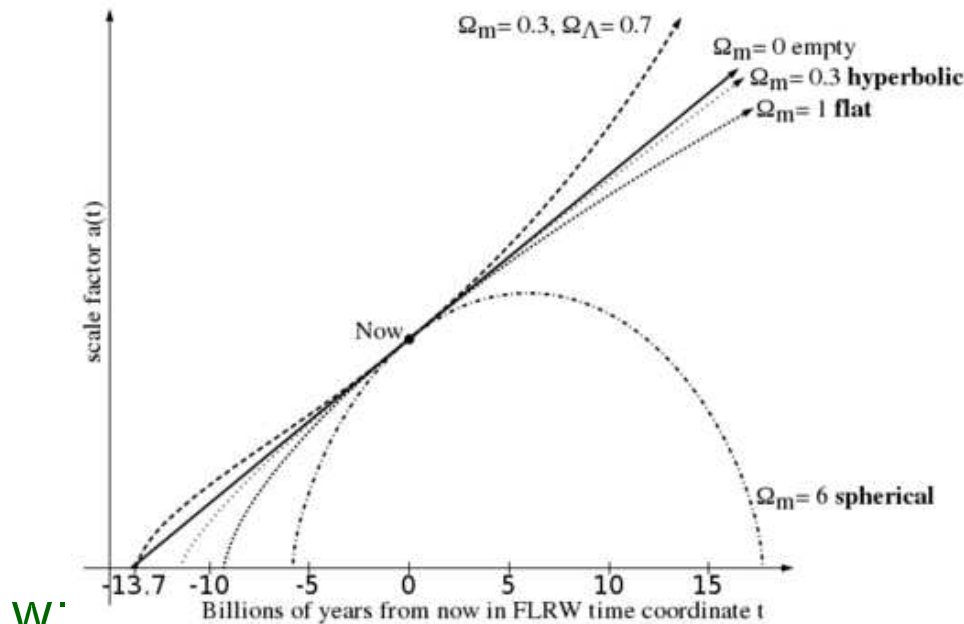
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(Defn:  $a_0 := 1$ )

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$$1 + z = \frac{1}{a_{\text{em}}}$$

(Defn of redshift  $z$ )

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- radiation density:  $E = h\nu \Rightarrow \rho_r \propto a^{-4} = (1+z)^4$

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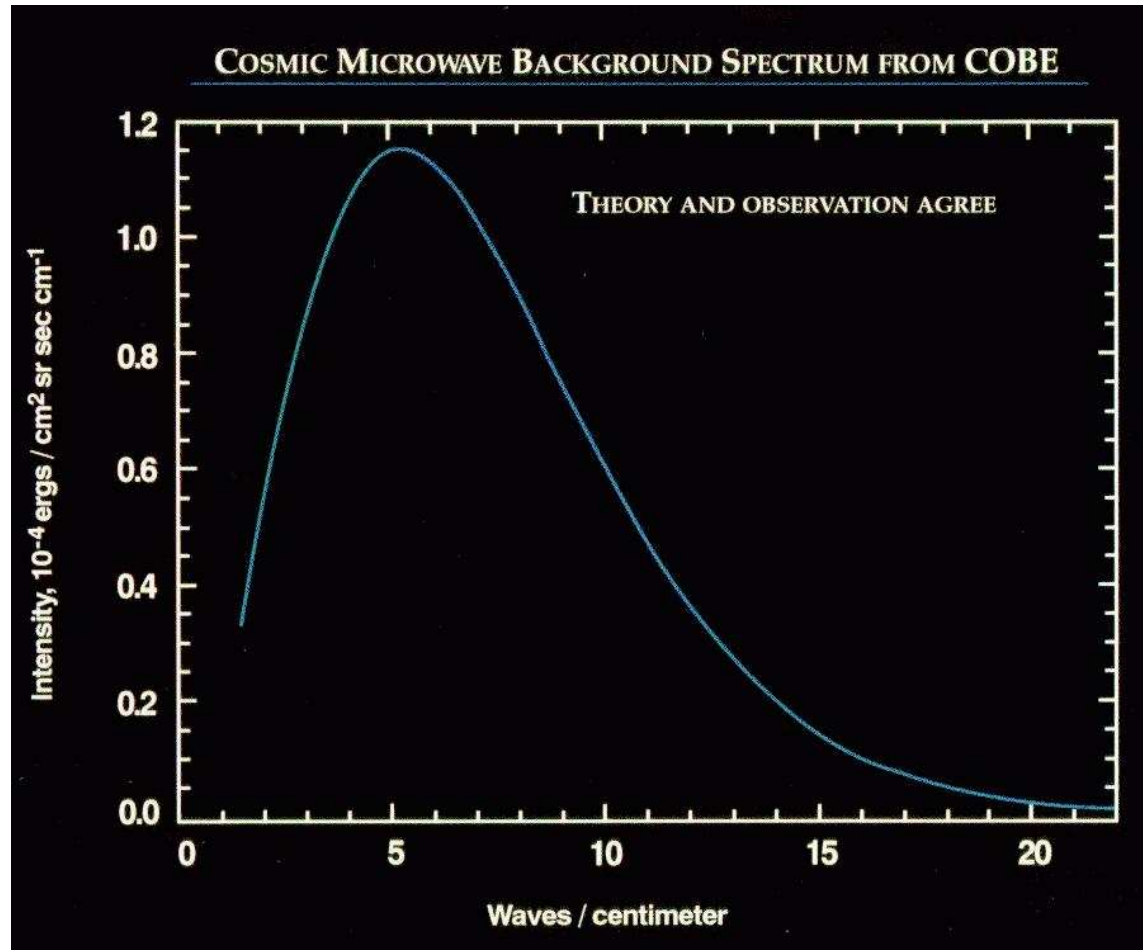
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- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)

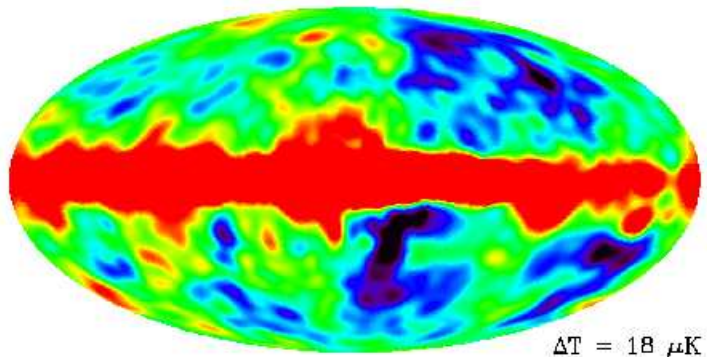
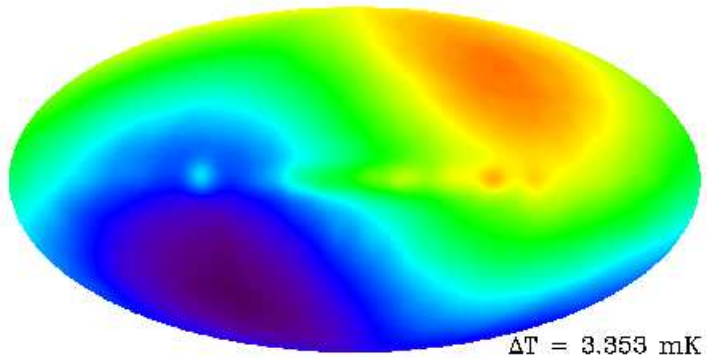
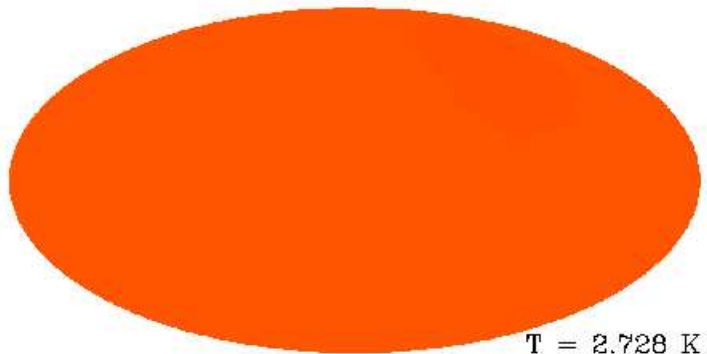
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- COBE /DMR (Differential Microwave Radiometer)



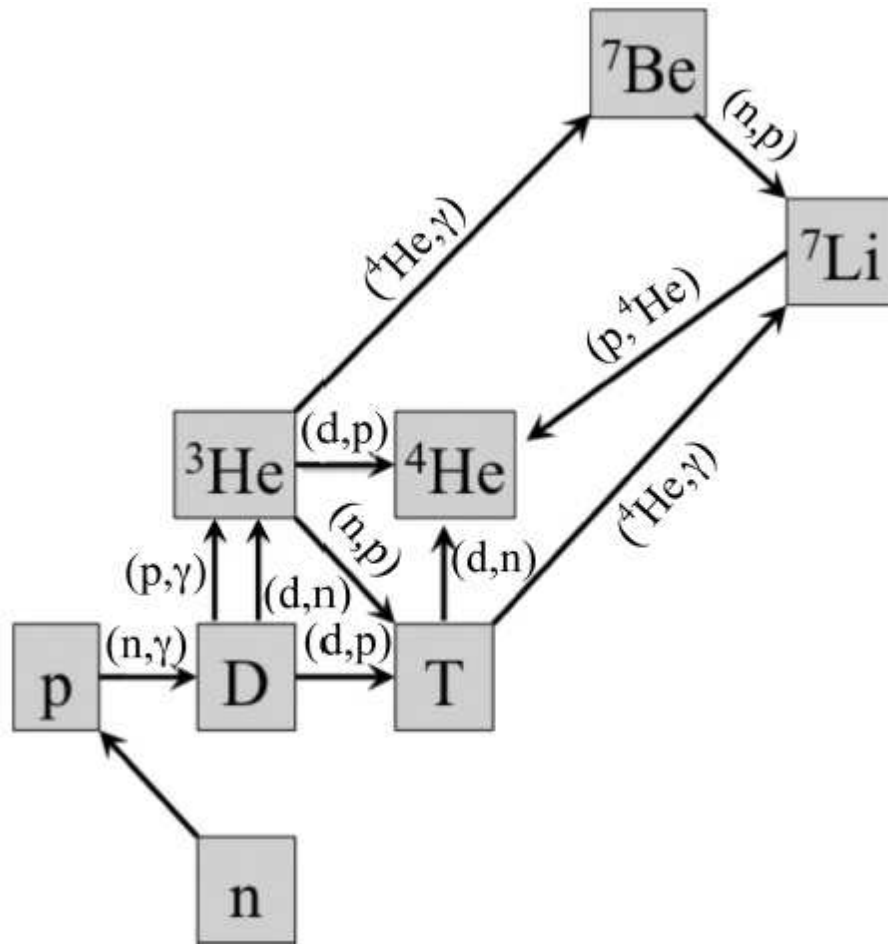
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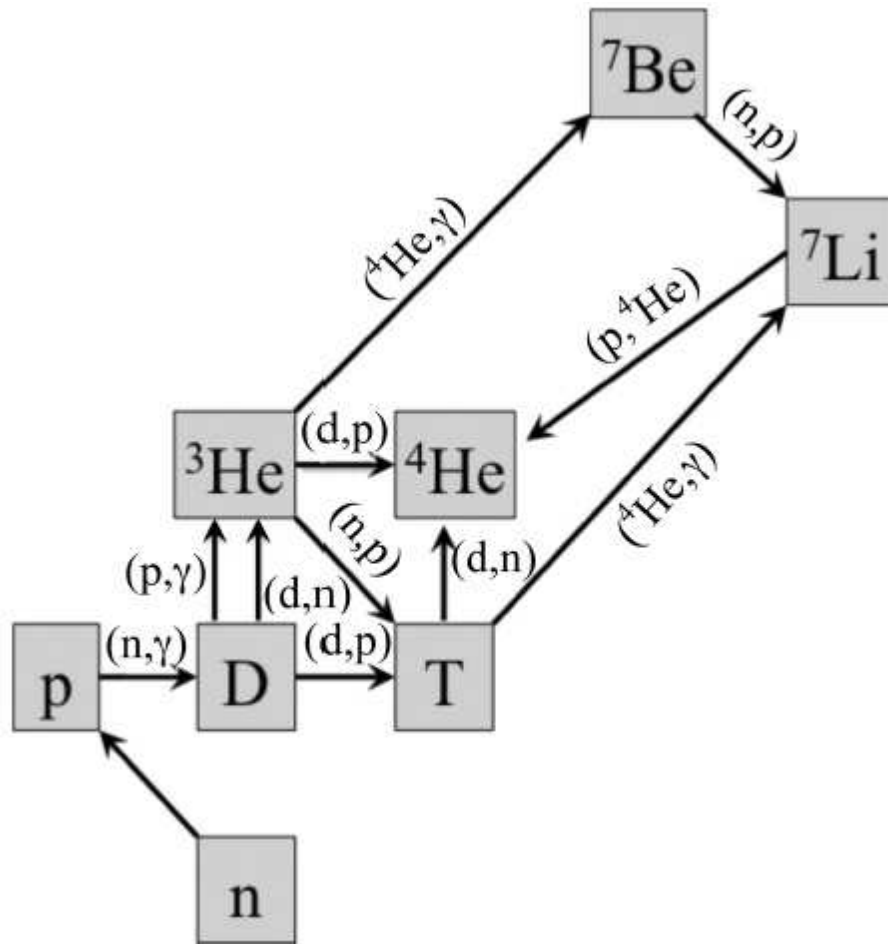
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- Friedmann Eqn: 
$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3}$$

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$$\text{Friedmann Eqn: } \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\pi G \rho}{3}$$

■

$$\text{acceleration Eqn: } \frac{\ddot{a}}{a} = -\frac{4\pi G (\rho + 3p/c^2)}{3}$$

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■ convenient conversion:  $1 \text{ km/s} \approx 1.04 \text{ kpc/Gyr} \approx 1 \text{ kpc/Gyr}$

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 curvature density parameter



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- consider a fixed observation, e.g.  $H_0 = 100 \text{ km/s/Mpc}$

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$$1 = \Omega_m + \Omega_k$$

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# FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords:  $R_C$



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■  $\Omega_{\text{tot}0} < 1$  *hyperbolic*  $R_C$  imaginary (or use  $|R_C|$ )

# Einstein's free parameter: $\Lambda$

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- *hint*: mixed index form of  $\mathbf{g}$  is easy

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Friedmann Eqn ( $\Lambda \neq 0$ ):

$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$



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Defn:  $q := -\frac{\ddot{a} a}{\dot{a}^2}$



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Defn: “dust solution”:  $p(t) = 0$

Defn:  $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

Defn:  $q := -\frac{\ddot{a}a}{\dot{a}^2}$  “deceleration parameter”

acceleration Eqn ( $\Lambda \neq 0$ ):

$$q = \frac{\Omega_m}{2} - \Omega_\Lambda$$

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# Einstein's free parameter: $\Lambda$

- Einstein: prevent expansion/contraction via  $\Lambda$   
ADS:1917SPAW.....142E

Friedmann Eqn ( $\Lambda \neq 0$ ):

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`cat myredshifts.lis | cosmdist`
- command line options: `cosmdist --help`
- static fortran or C library: link to `libcosmdist.a`

# distances in FLRW cosmology

- $r = \int_{1/(1+z)}^1 \frac{c}{H_0} \frac{da}{a \sqrt{\Omega_{m0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$
- GPL numerical package: `cosmdist`  
<http://cosmo.torun.pl/GPLdownload/cosmdist/>
- standard install to `/usr/local`:  
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- high-level frontends (e.g. python) should be easy to write

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- $d_A = r_{\perp} a$  (scaled by the scale factor)



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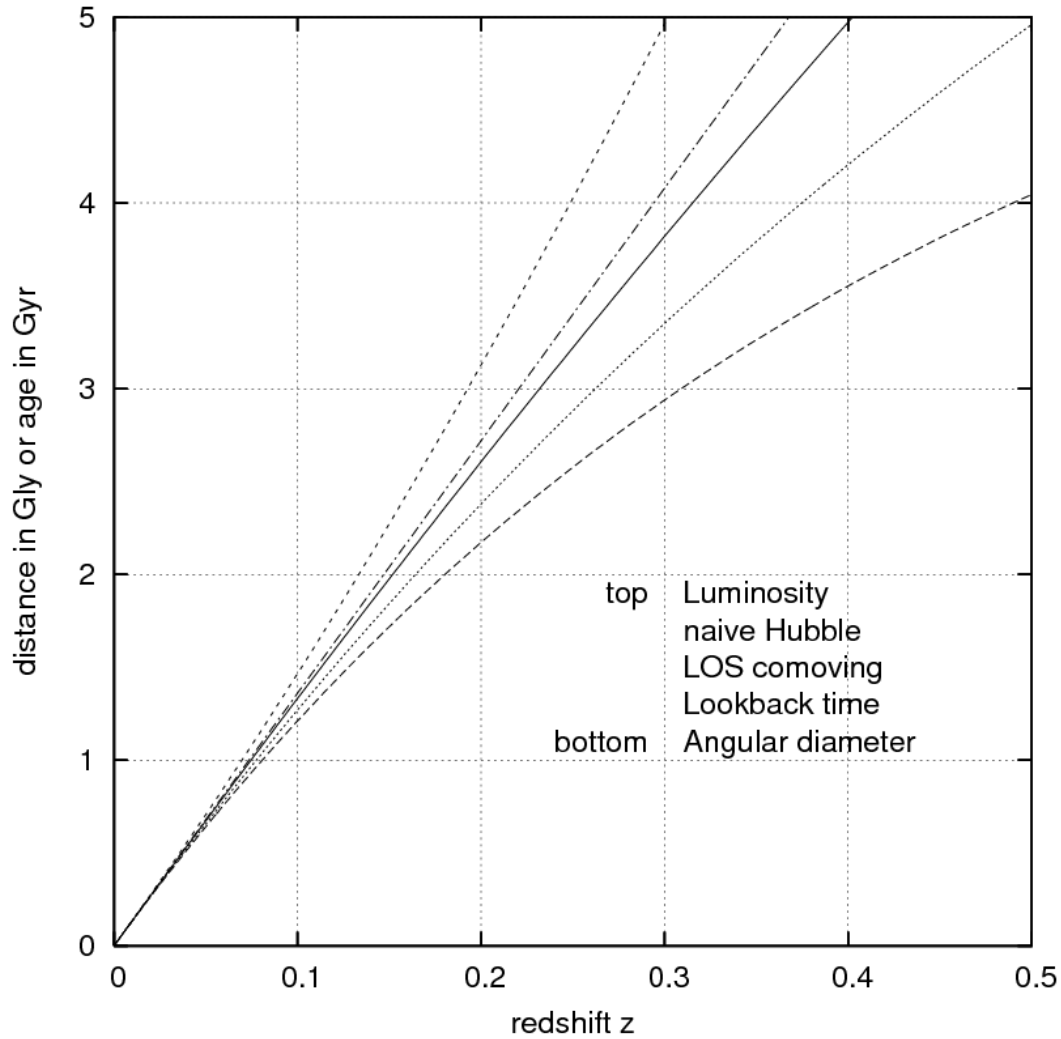
- w:Distance measures (cosmology)



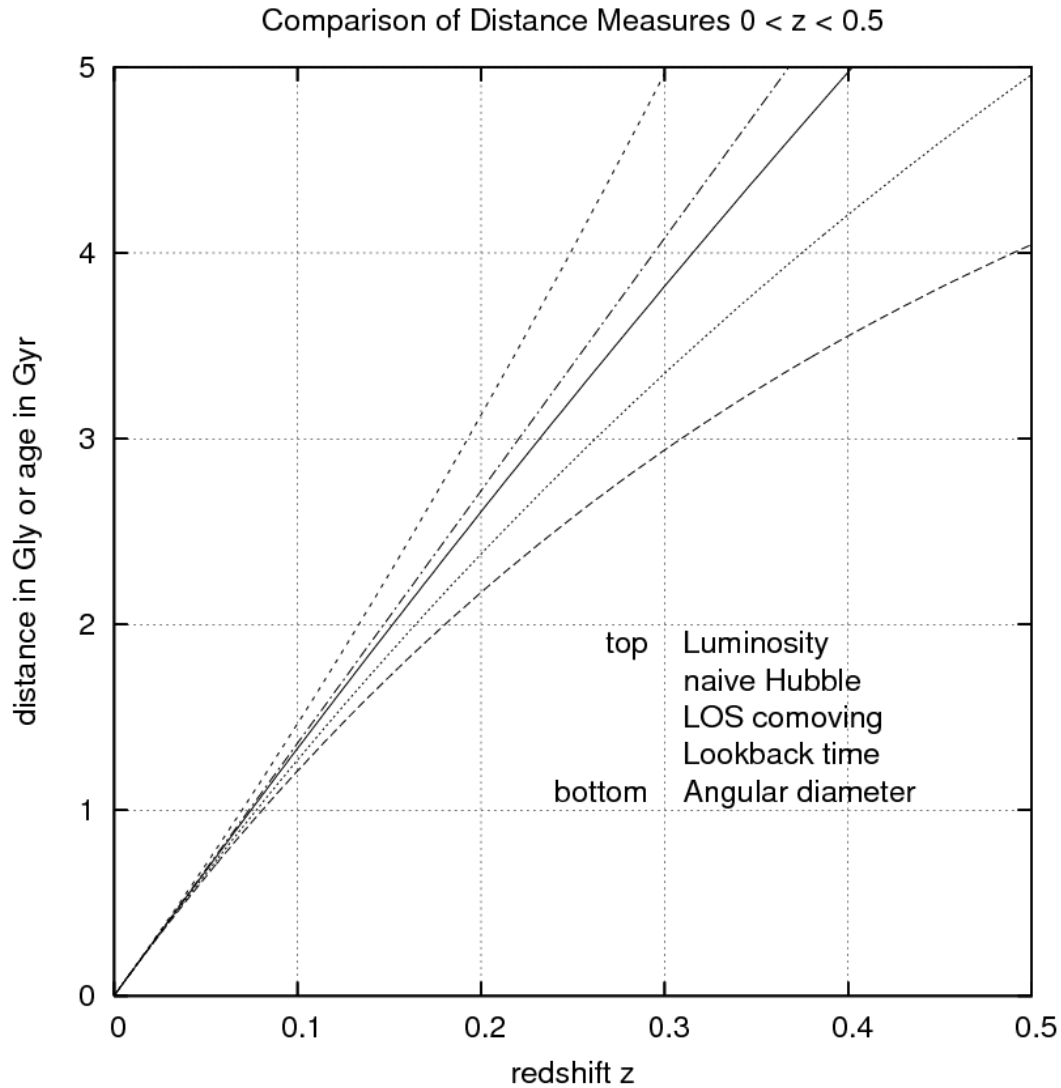
# FLRW distances e.g. $\Lambda$ CDM



Comparison of Distance Measures  $0 < z < 0.5$

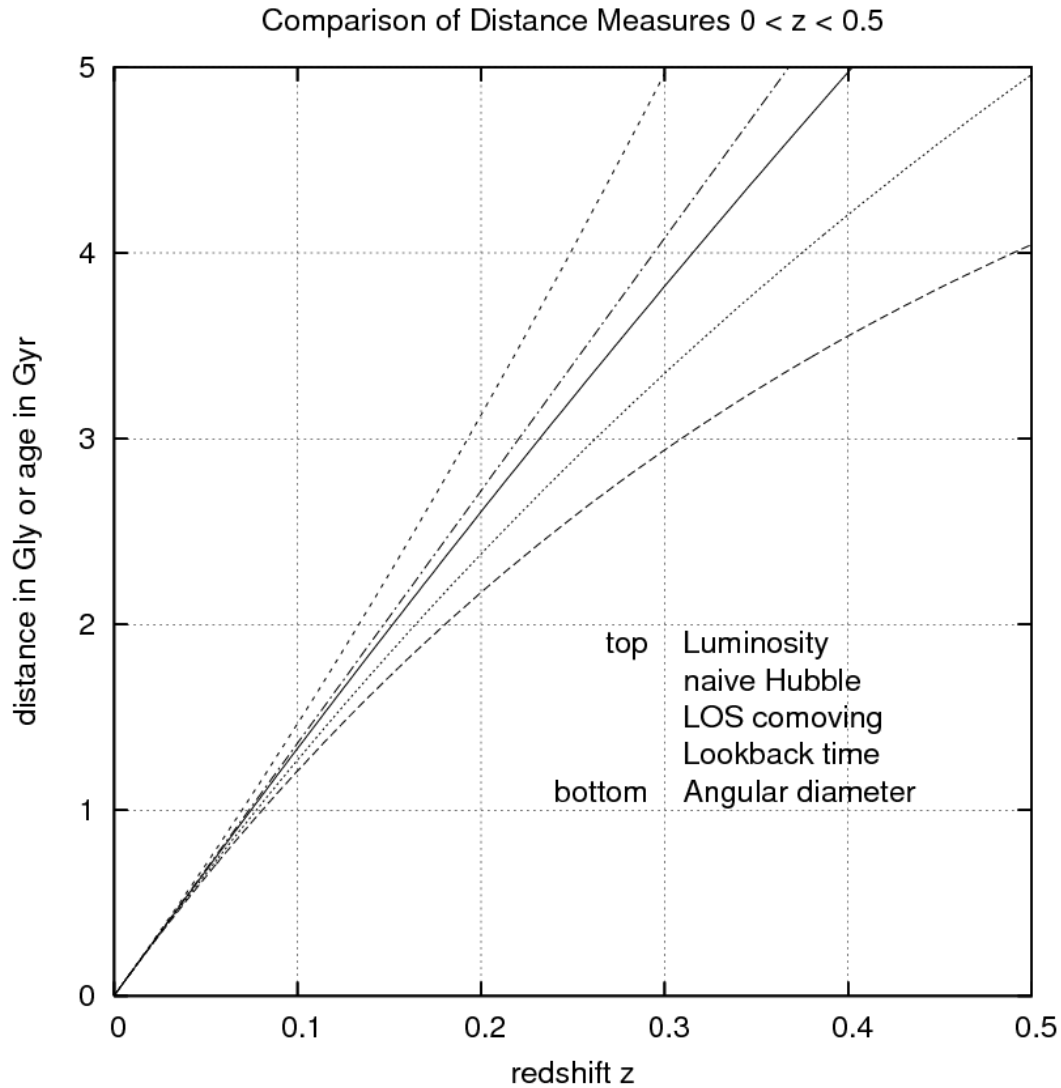


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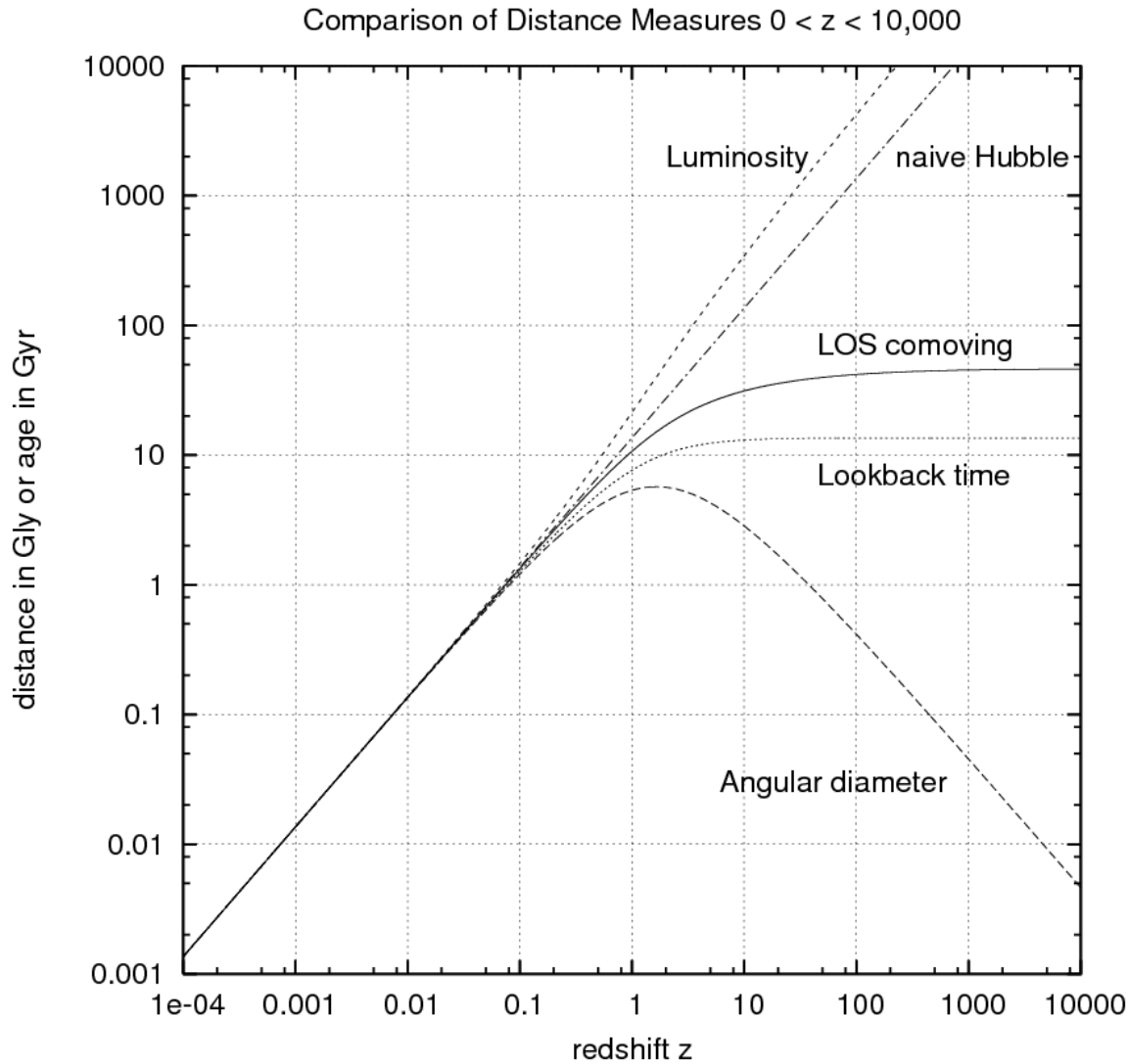
Defn:  $h := H_0/100 \text{ km/s/Mpc}$  (without a "0" subscript on  $h$ )

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$\Rightarrow$  no conflict with locally Lorentzian (SR) spacetime

# Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature (+, 0, -)?

# Non-radial spatial geodesics

- distances on the 2-sphere, embedded in  $\mathbb{R}^3$

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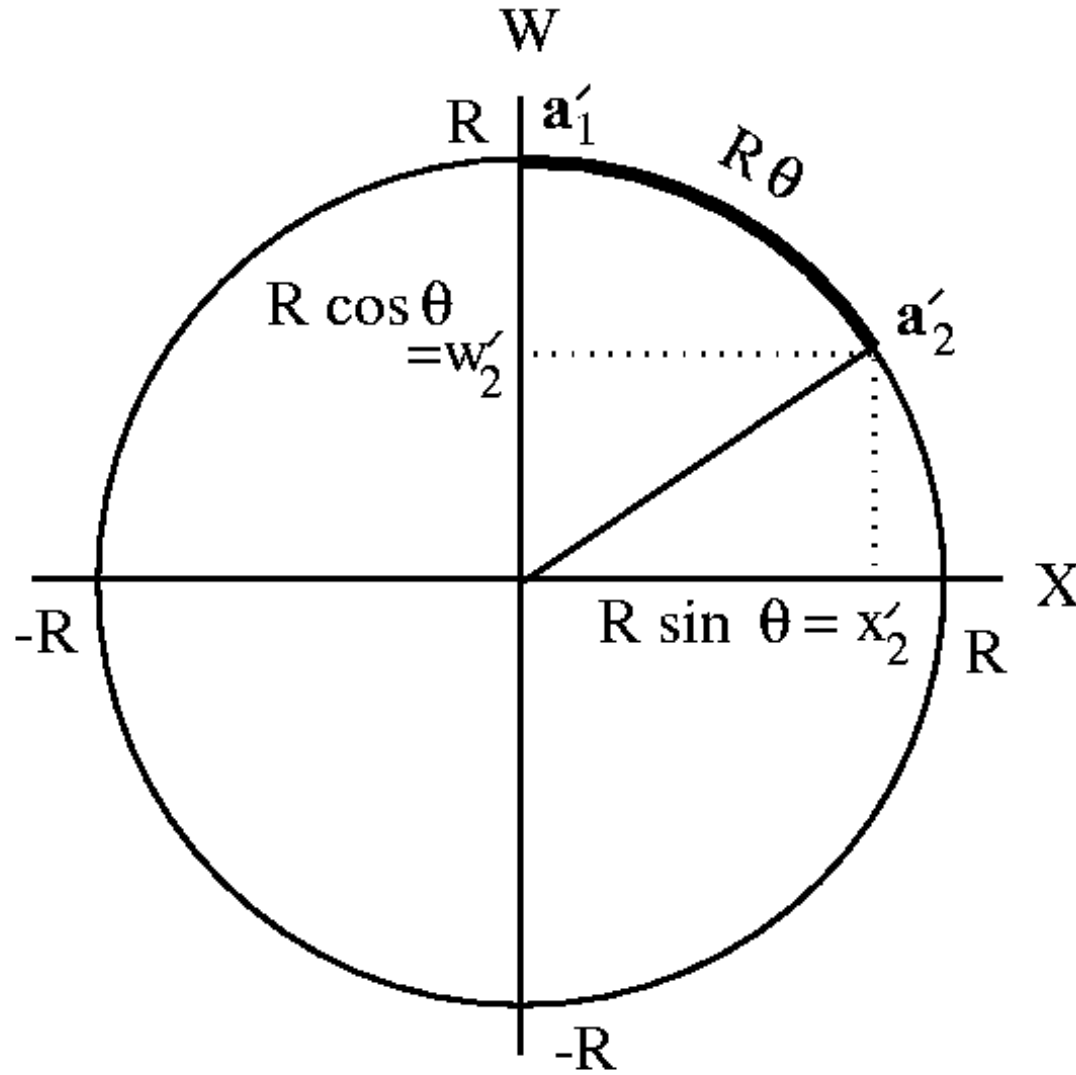
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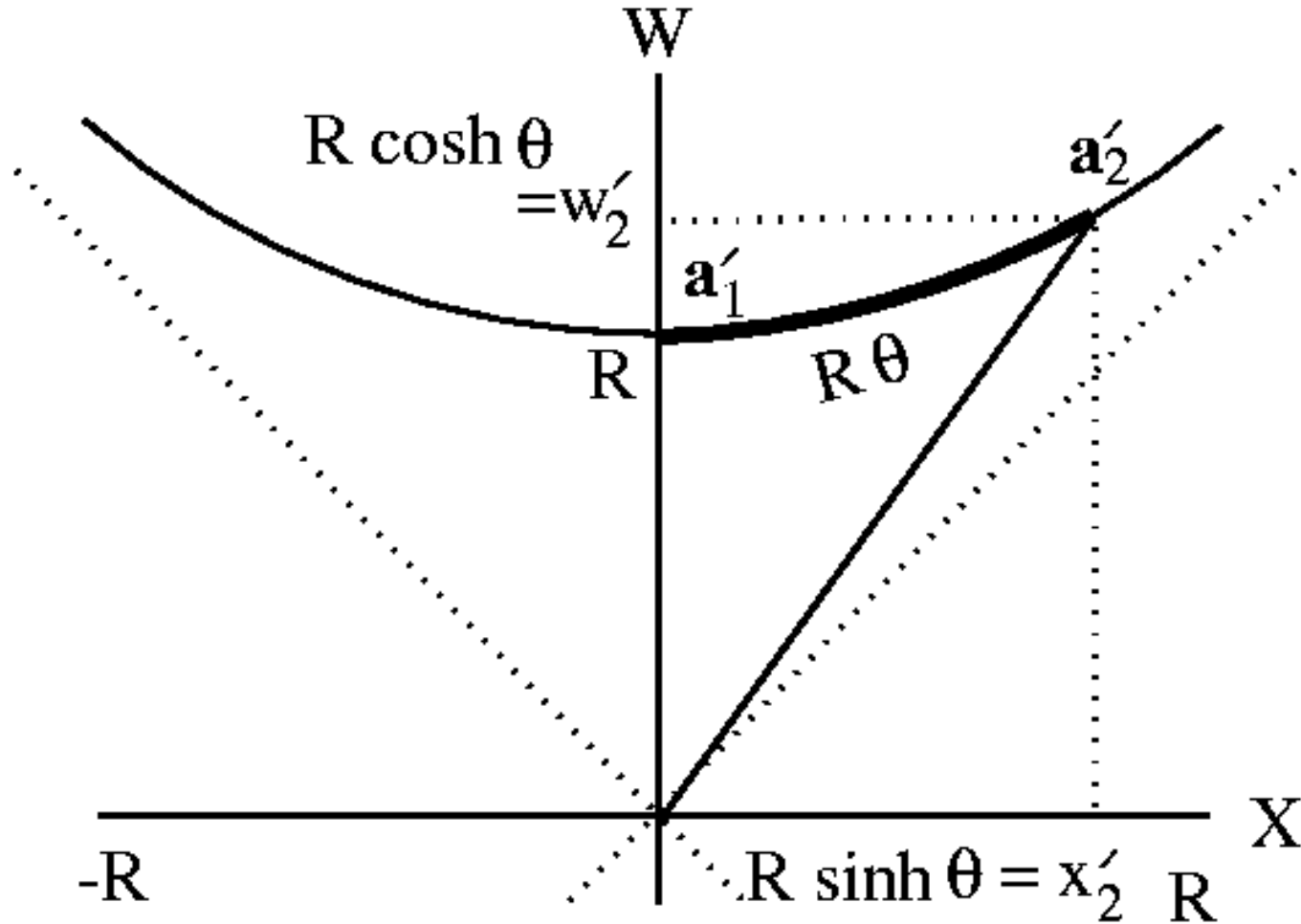


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metric on  $S^3$  (or  $\mathbb{R}^3$  or  $H^3$ ):

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- Jan Ostrowski pdf 21.02.2013 (unpublished)