Cosmology

B.F. Roukema

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Introduction

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- verbal averaging: can we do better?

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- but: $\exists$ galaxies ⇒ inhomogeneous, anisotropic spatial slices
- standard model: density perturbations (anisotropy)
- scalar (GR) averaging: statistically homogeneous spatial slices

verbal averaging

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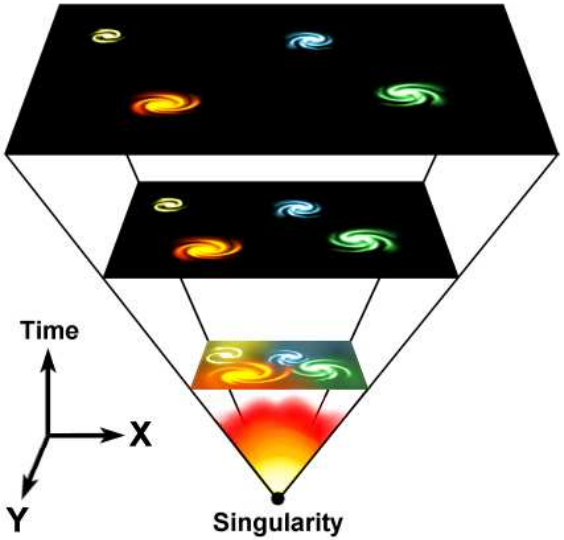
1. assume homogeneity and isotropy
2. find the (differential 4-pseudo-manifold, metric) pairs $(M, \mathbf{g})$ that solve $\mathbf{G} = 8\pi\mathbf{T}$
3. assume that $(M, \mathbf{g})$ remains unchanged if we add density perturbations to an early time slice

verbal averaging

- w:Comoving coordinates

verbal averaging



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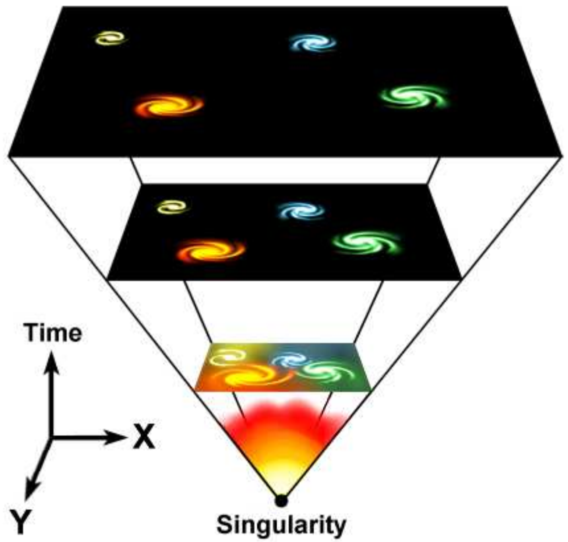

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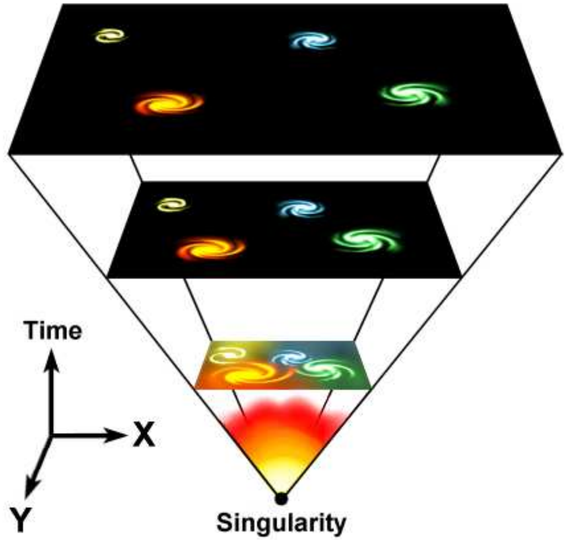
$$\Delta x(t) = a(t)\Delta r$$

verbal averaging



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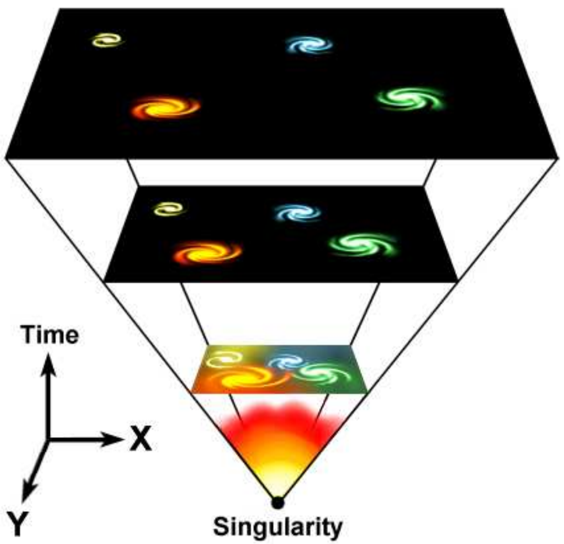
- spherical coordinates for spatial slice

verbal averaging



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$$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} \mathrm{d}s = a(t)\Delta r = a(t)|r_2 - r_1|$$

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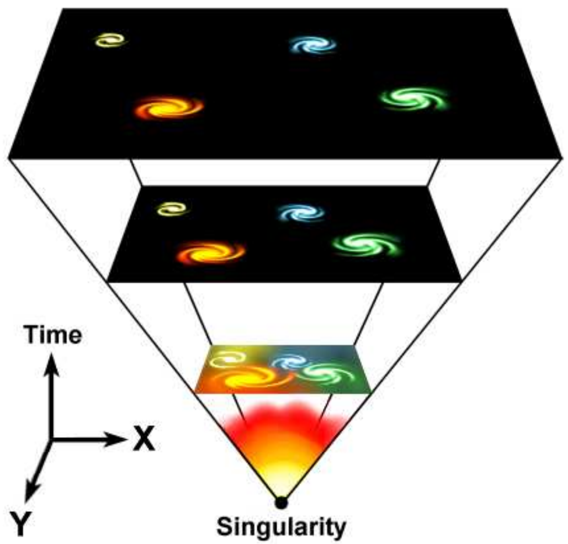

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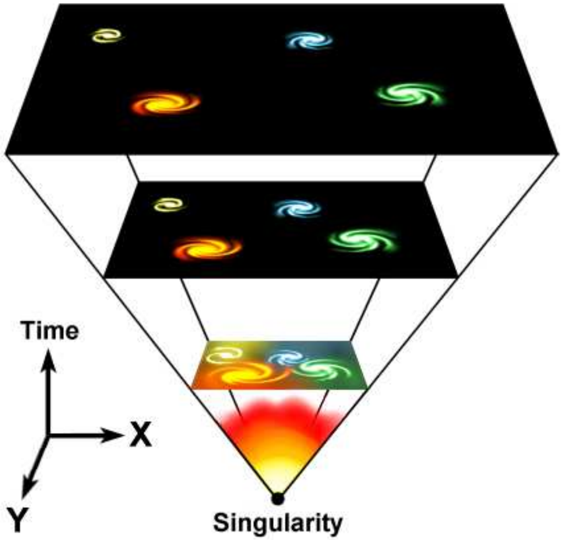

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- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

- w:Friedmann–Lemaître–Robertson–Walker metric

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- w: w:
  w:Howard Percy Robertson
  w:Arthur Geoffrey Walker

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ldots$$

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$$u := \int \frac{\mathrm{d}t}{a(t)}$$

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$$ds^2 = a^2(u)\left[-du^2 + dr^2 + r_\perp^2(d\theta^2 + \cos^2\theta d\phi^2)\right]$$

- but $\int du \neq$ proper time; *more:* arXiv:astro-ph/0707.2106

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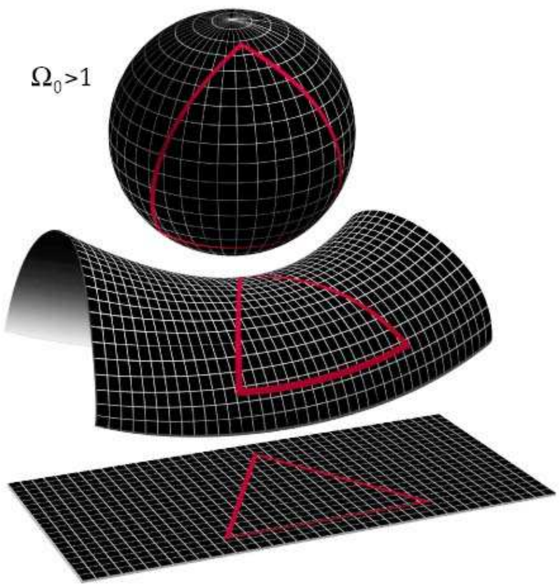
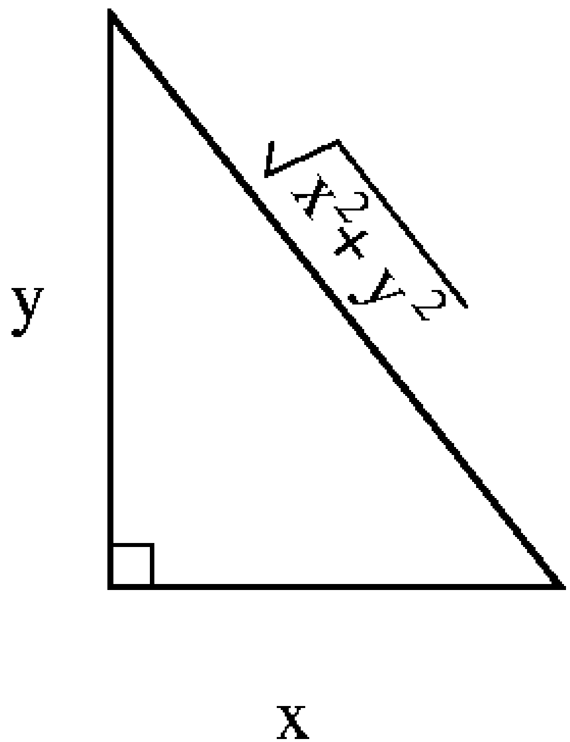


where $$r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k>0 \end{cases}$$

for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):



$$k = 0$$

curvature

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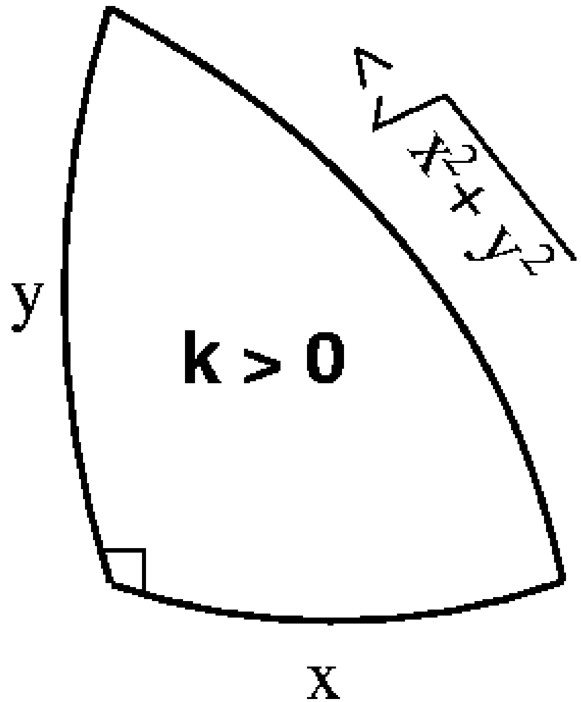

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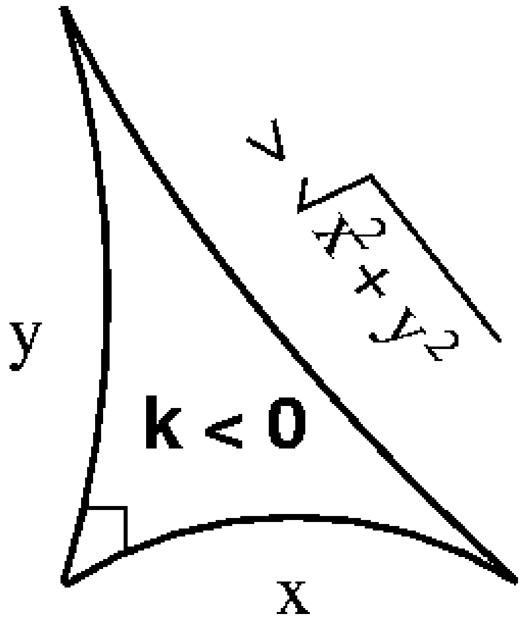

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2D curvature intuition: $k > 0$

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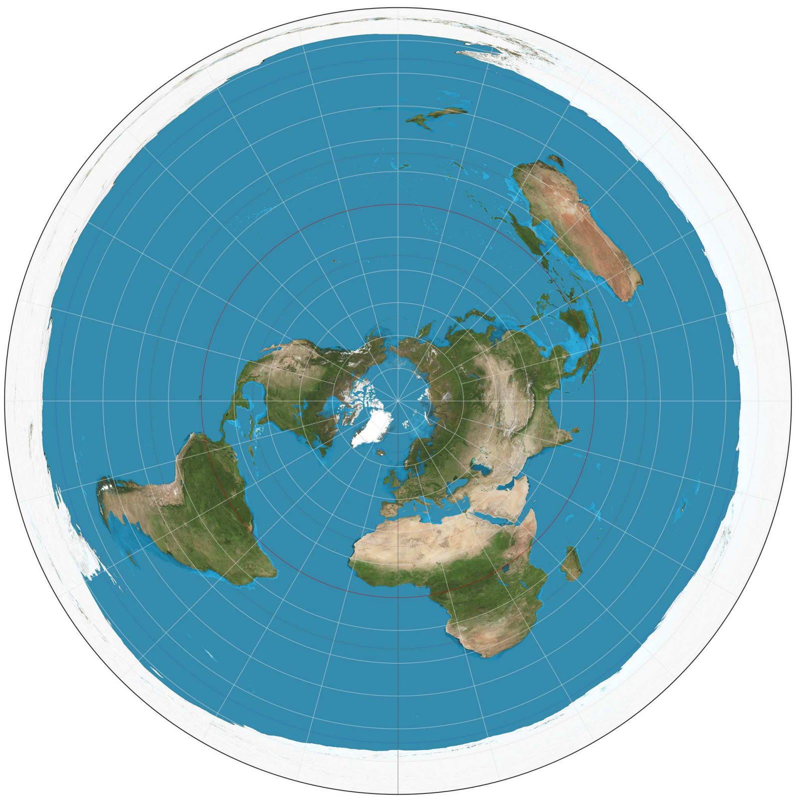

w: (al-Biruni, c. 1000 CE)

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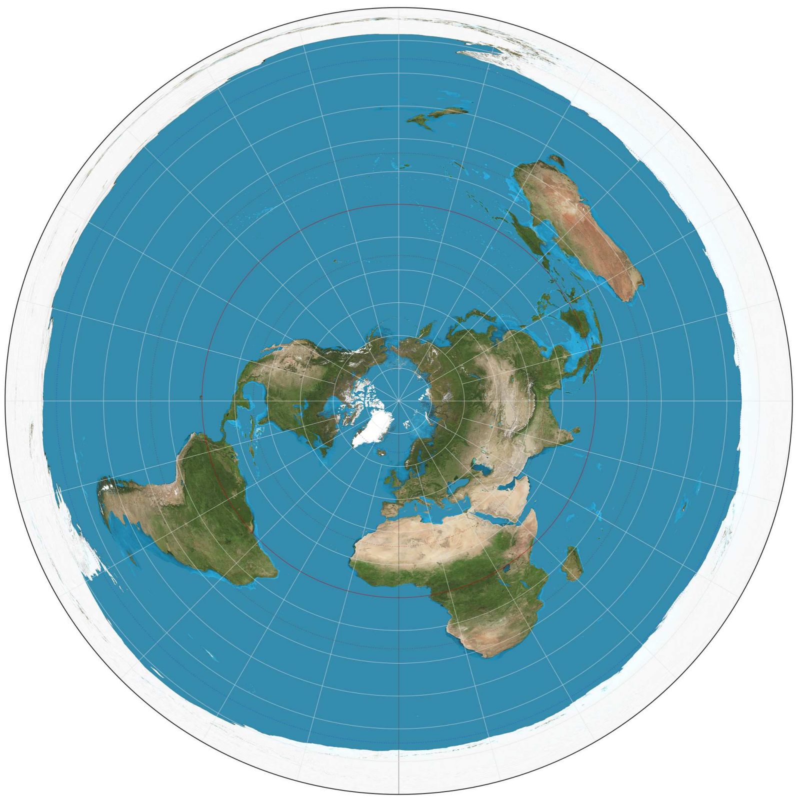

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- intuition switch: S^2 easier vs S^3 more physical

2D topology intuition ($k = 0$)

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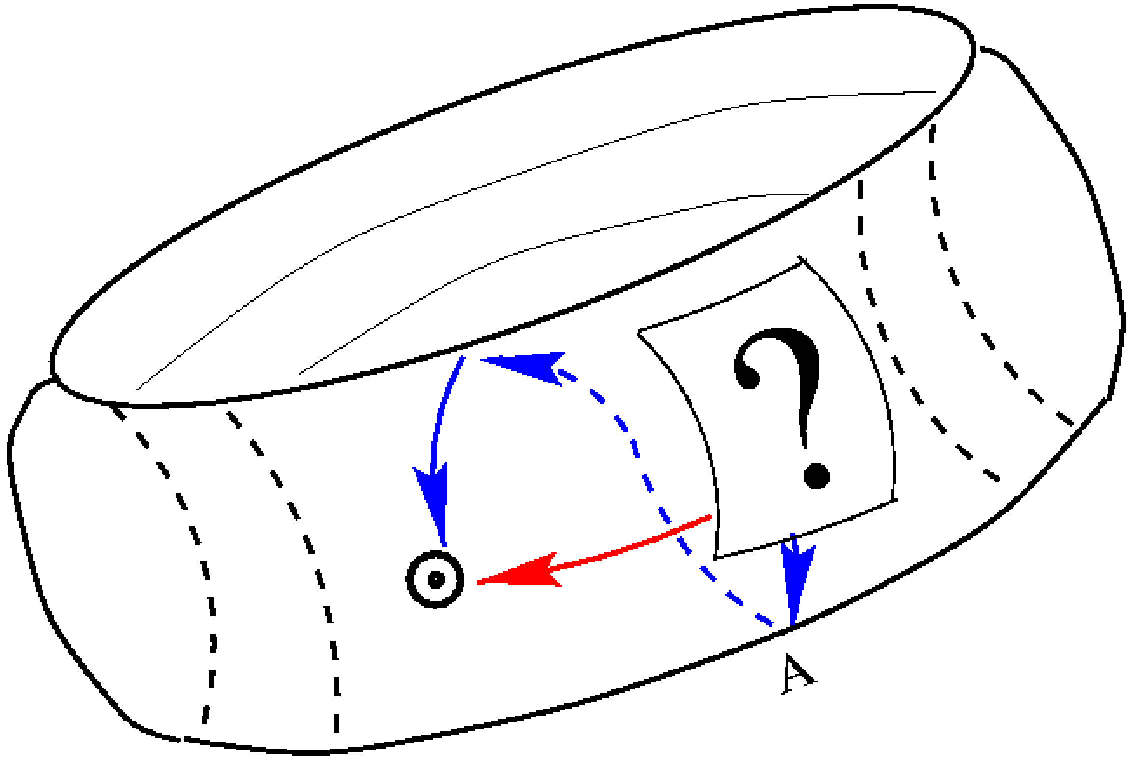

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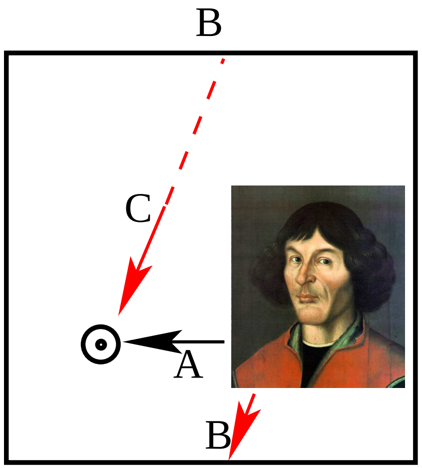

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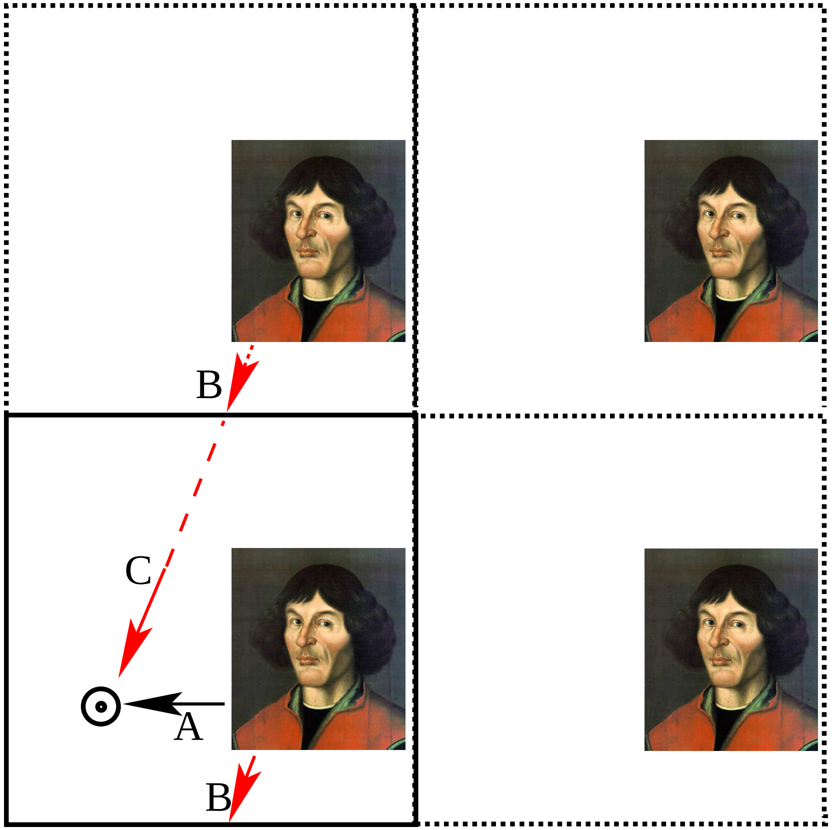

expansion

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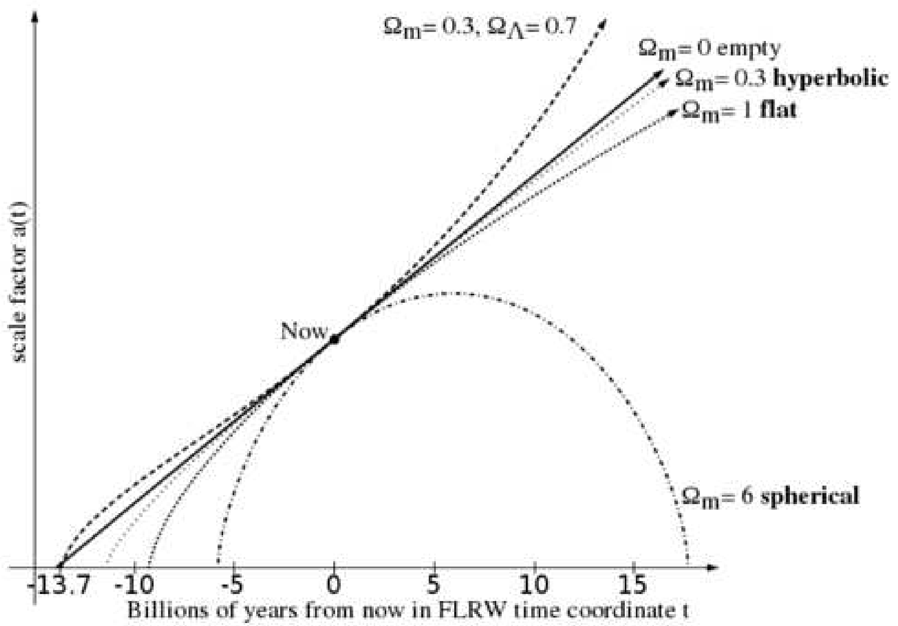

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$\Leftrightarrow$

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  (Defn: $a_0 := 1$)

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$$1 + z = \frac{1}{a_{\rm em}}$$

(Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_{\mathrm{r}} \propto a^{-4} = (1+z)^4$

Black body: COBE (~ 1992)

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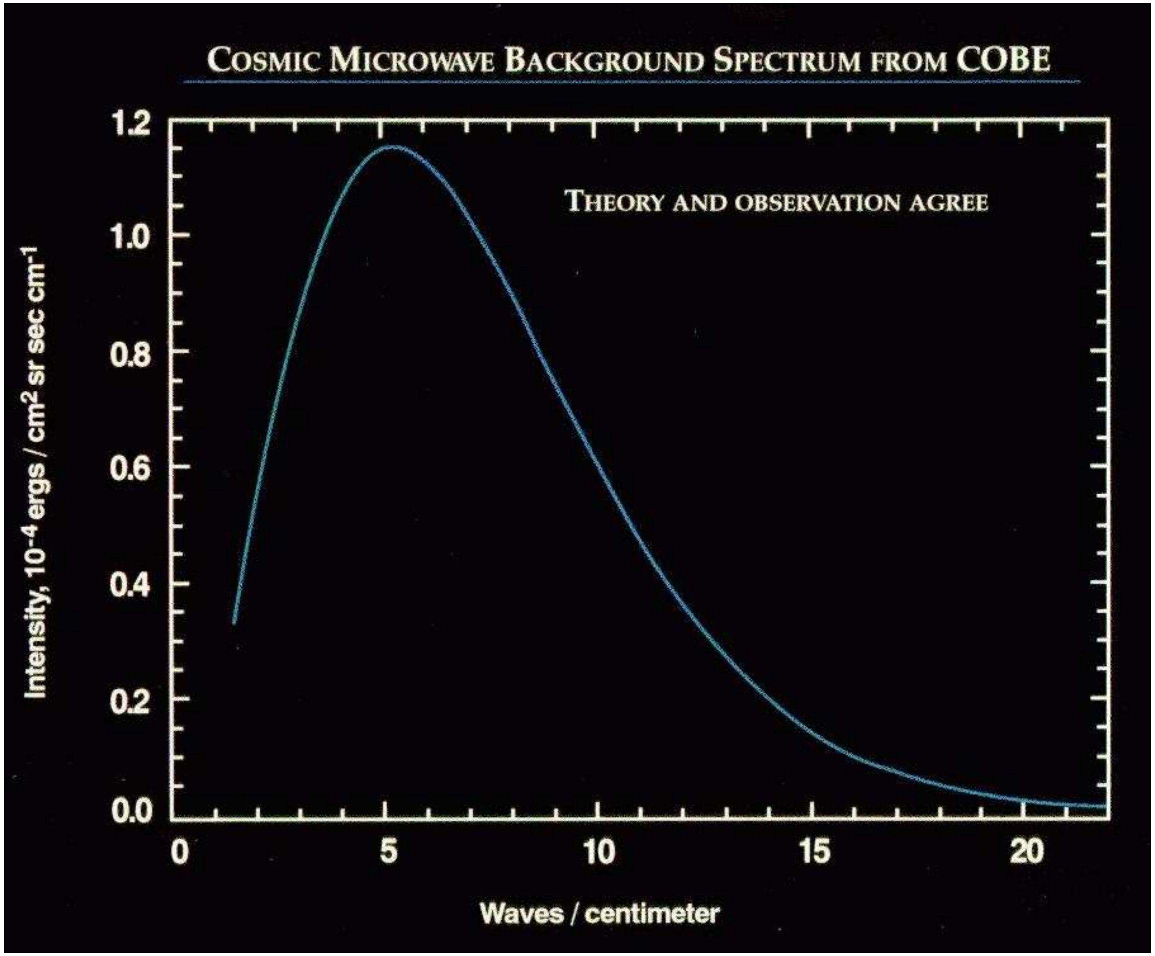

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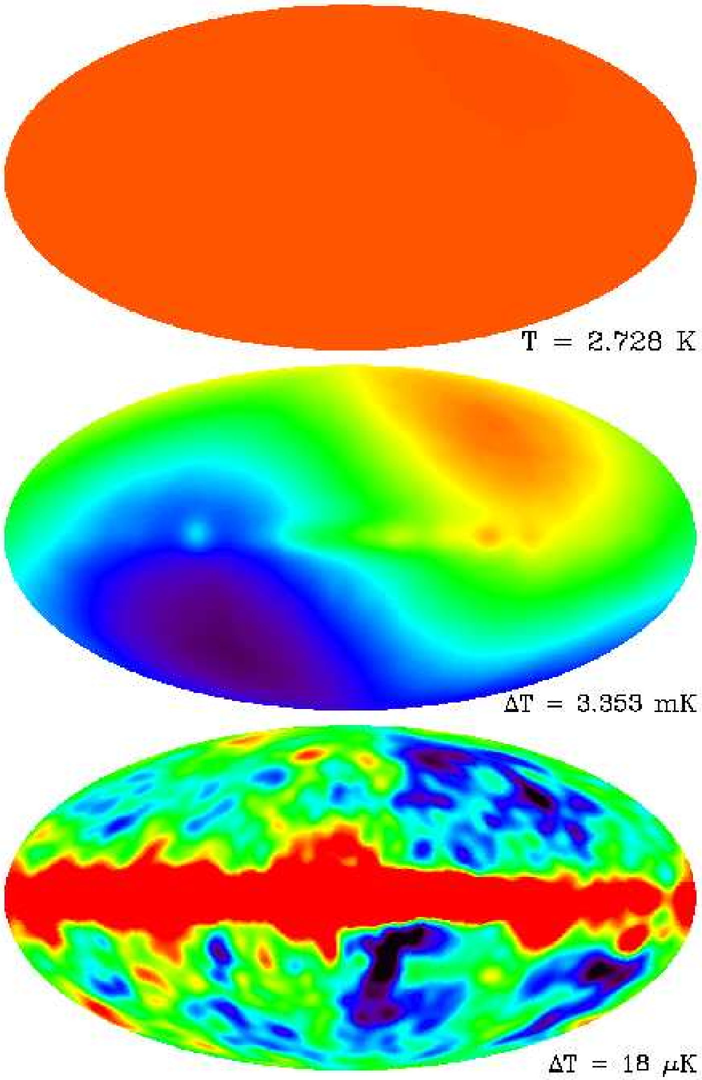
- COBE /DMR (Differential Microwave Radiometer)



BBN: Big bang nucleosynthesis

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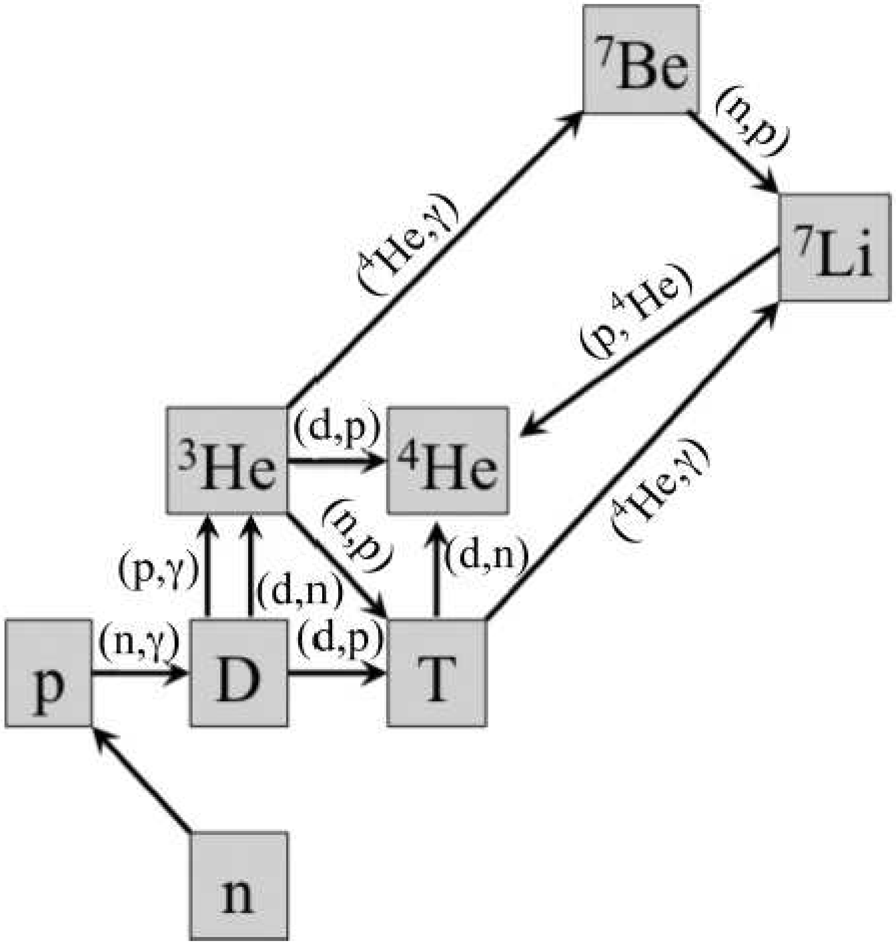

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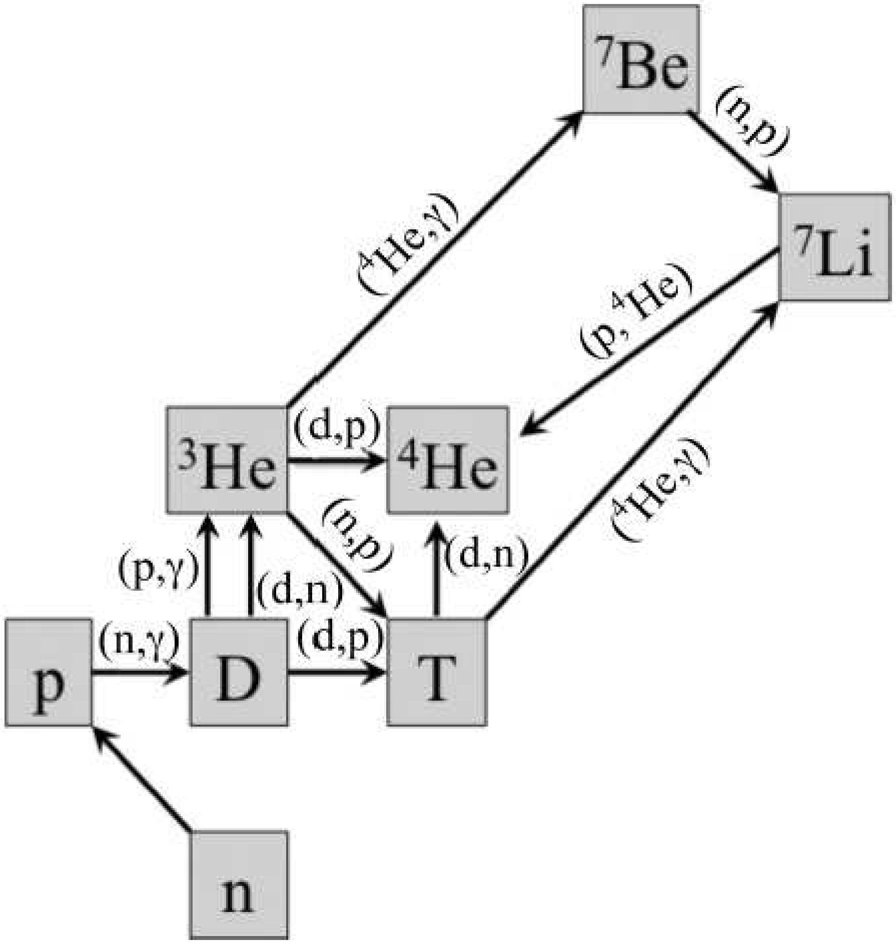

w:Big Bang nucleosynthesis

FLRW: $a(t) = ?$

- second choice FLRW coord system: $r :=$ w:orthographic projection of radial comoving distance (cf $r :=$ radial comoving distance)

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- acceleration Eqn: $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$

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- matter-dominated epoch: $\rho = \rho_{\rm m}$

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FLRW matter-dominated epoch

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- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

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- $k = 0$ case: $\mathrm{d}a \propto a^{-1/2}\mathrm{d}t$ for $a > 0$

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- $k = 0$ case: $a^{1/2}\mathrm{d}a \propto \mathrm{d}t$

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- $k = 0$ case: $(2/3)a^{3/2} \propto t$

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- $k = 0$ case: $a \propto t^{2/3}$

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- $k = 0$ case: $$\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$$ Einstein–de Sitter model (EdS)

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$\Rightarrow$ $H(t) = \frac{2}{3t}$;

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- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0}$

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- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

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  - $\rho_{\rm m0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$ hyperbolic

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Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

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Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter

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Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 + \Omega_k H^2$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat

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  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} :=$

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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm m} +$

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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$\boxed{1 = \Omega_{\rm tot} + \Omega_k}$$

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  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\rm tot} := \Omega_{\rm b} + \Omega_{\rm nbDM} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_C
  - orthographic coords: k

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

FLRW curvature constant

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k0} H_0^2}{c^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

FLRW curvature constant

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{\Omega_{k0}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $$\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$$

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $$\boxed{R_{\rm C} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\rm tot0} - 1}}}$$
- $\Omega_{\rm tot0} > 1$ *spherical*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $$R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}$$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$

FLRW curvature constant

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  - orthographic coords: k
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic*

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* R_{C} imaginary (or use $|R_{\mathrm{C}}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- *hint:* mixed index form of $\mathbf{g}$ is easy

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{4 \pi G \left(\rho + 3 p/c^2\right)}{3} + \Omega_\Lambda H^2$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$):
$$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\text{crit}}} + \Omega_\Lambda\,H^2$$

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_{\mathrm{m}}}{2} + \Omega_\Lambda\,H^2$$

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Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

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Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda$$

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- $\boxed{q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Lambda = 0$ and $\Omega_\mathrm{m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Lambda = 0$ and $\Omega_{\rm m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$ *deceleration*

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Omega_\Lambda > \Omega_{\rm m}/2$ then $\frac{\ddot{a}}{a} > 0$, i.e. $q < 0$ *acceleration*

distances in FLRW cosmology

- azimuthal equidistant r:

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$\Rightarrow \dot{a}^2 = H_0^2\,(\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2)$

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$\Rightarrow \mathrm{d}a = H_0 \sqrt{\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}\, \mathrm{d}t$ if $\dot{a}, a, H_0 > 0$

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Low z limit Hubble law

- use r = comoving radial distance

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0}$

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $\boxed{r \approx \frac{z\,c}{H_0} \approx \frac{\beta\,c}{H_0} = \frac{v}{H_0}}$

EdS: radial comoving distance

- $r = \int_{1/(1+z)}^{1} \frac{c}{H_0} \frac{\mathrm{d}a}{a\sqrt{\Omega_{\mathrm{m}0}a^{-1}+\Omega_{k0}+\Omega_{\Lambda 0}a^2}}$

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&= \frac{c}{H_0} \left[2a^{1/2}\right]_{1/(1+z)}^{1} \\
&= \boxed{\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)} \\
&\approx 6h^{-1}\mathrm{Gpc}\left(1 - \frac{1}{\sqrt{1+z}}\right)
\end{aligned}
$$

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  `./configure && make && make install`

- command line pipe:
  `cat myredshifts.lis | cosmdist`

- command line options: `cosmdist --help`

- static fortran or C library: link to `libcosmdist.a`

- high-level frontends (e.g. python) should be easy to write

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want to define d_L such that flux is $F = \dfrac{L}{4\pi d_L^2}$

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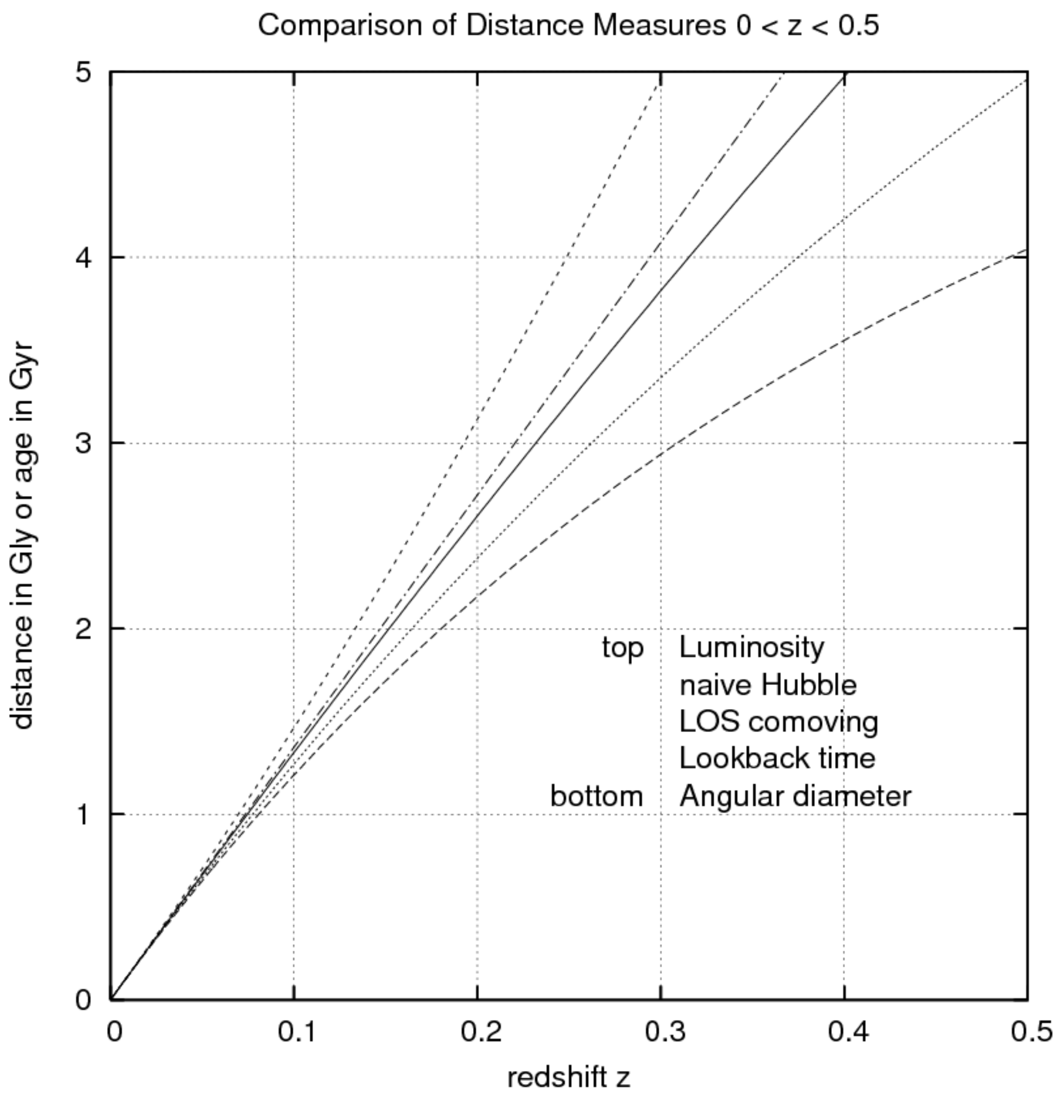
- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM



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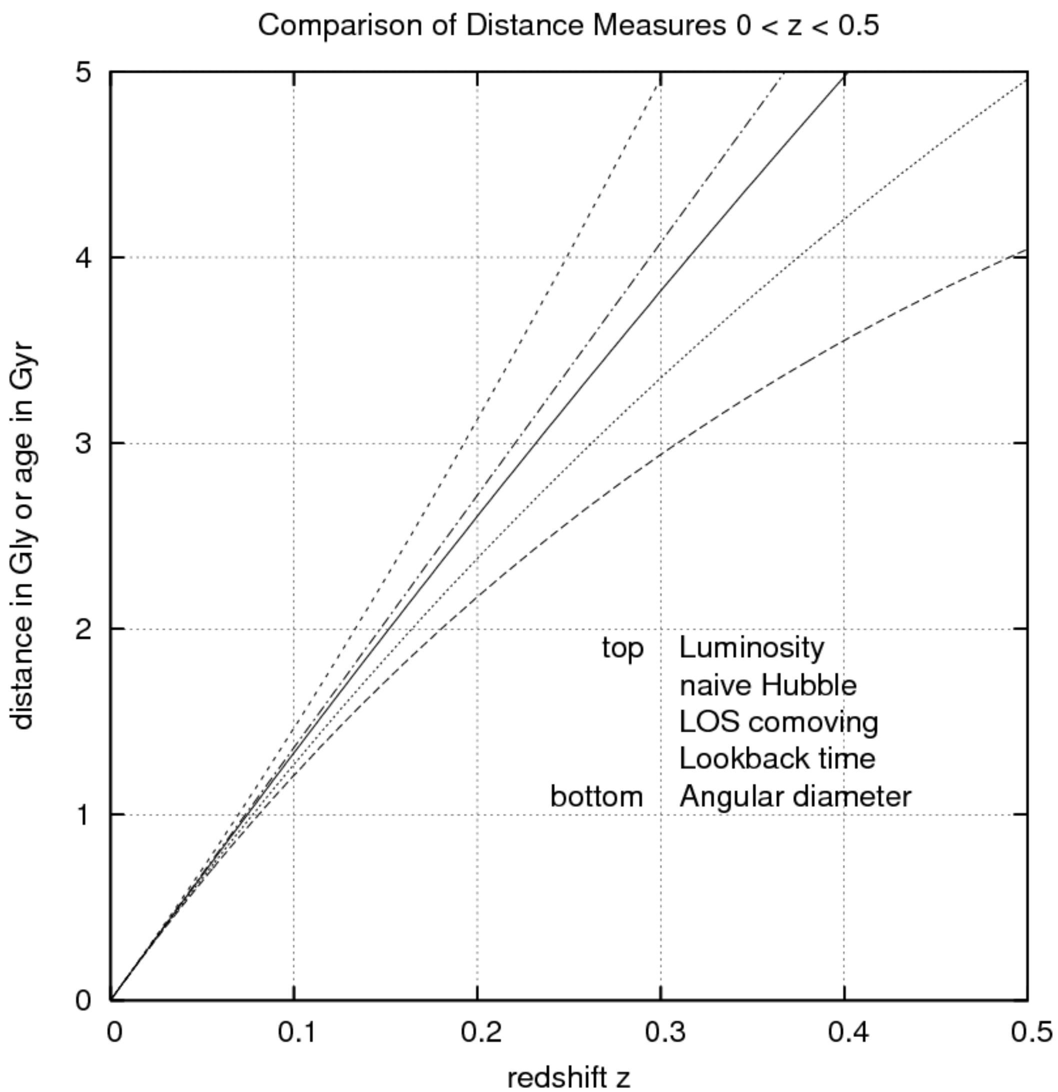

Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

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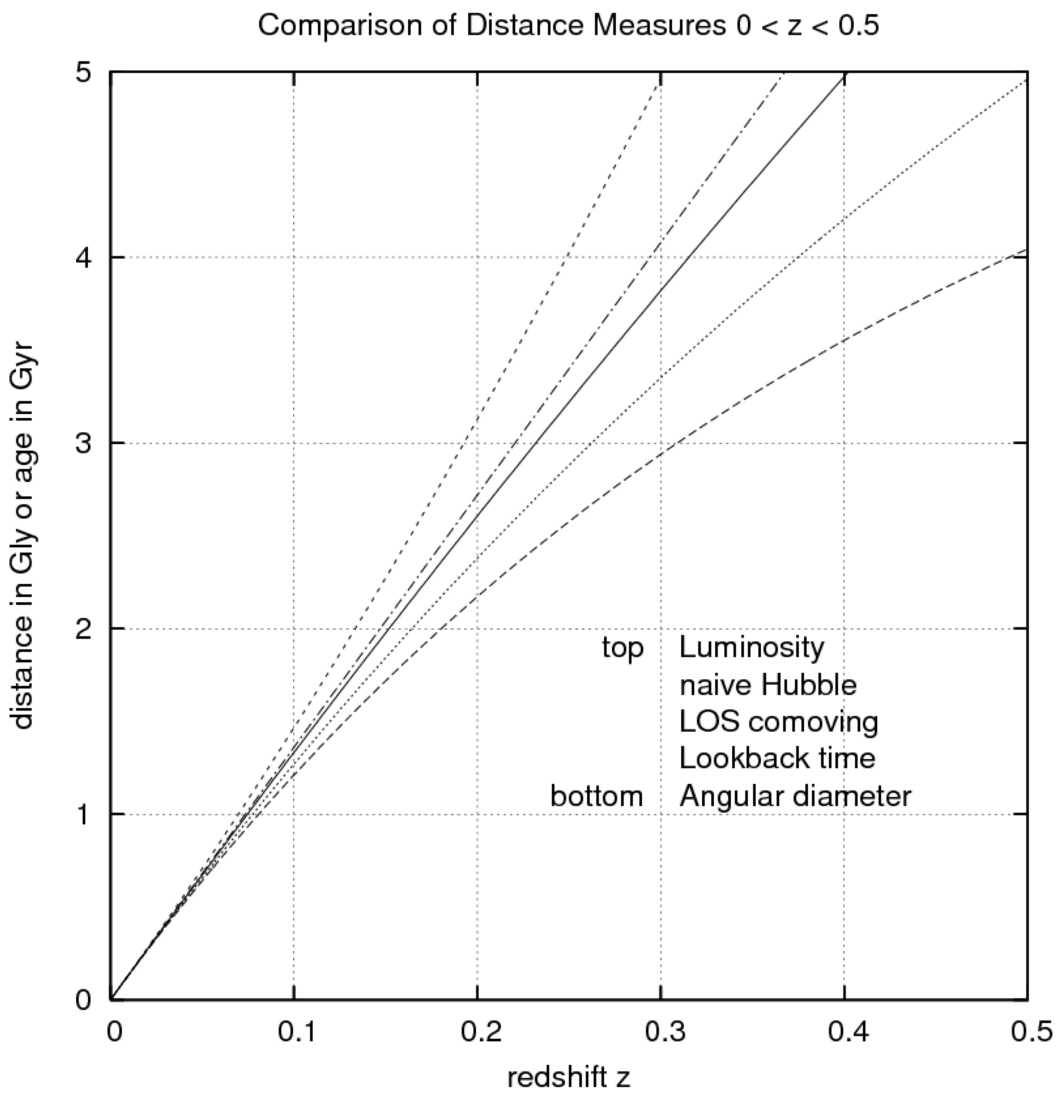

$$\Omega_{\rm m0} = 0.3,\ \Omega_{\rm r0} = 10^{-4},\ \Omega_{\Lambda 0} = 1.0 - (\Omega_{\rm m0} + \Omega_{\rm r0}),\ h = 0.7,\ \Omega_{k0} = 0$$

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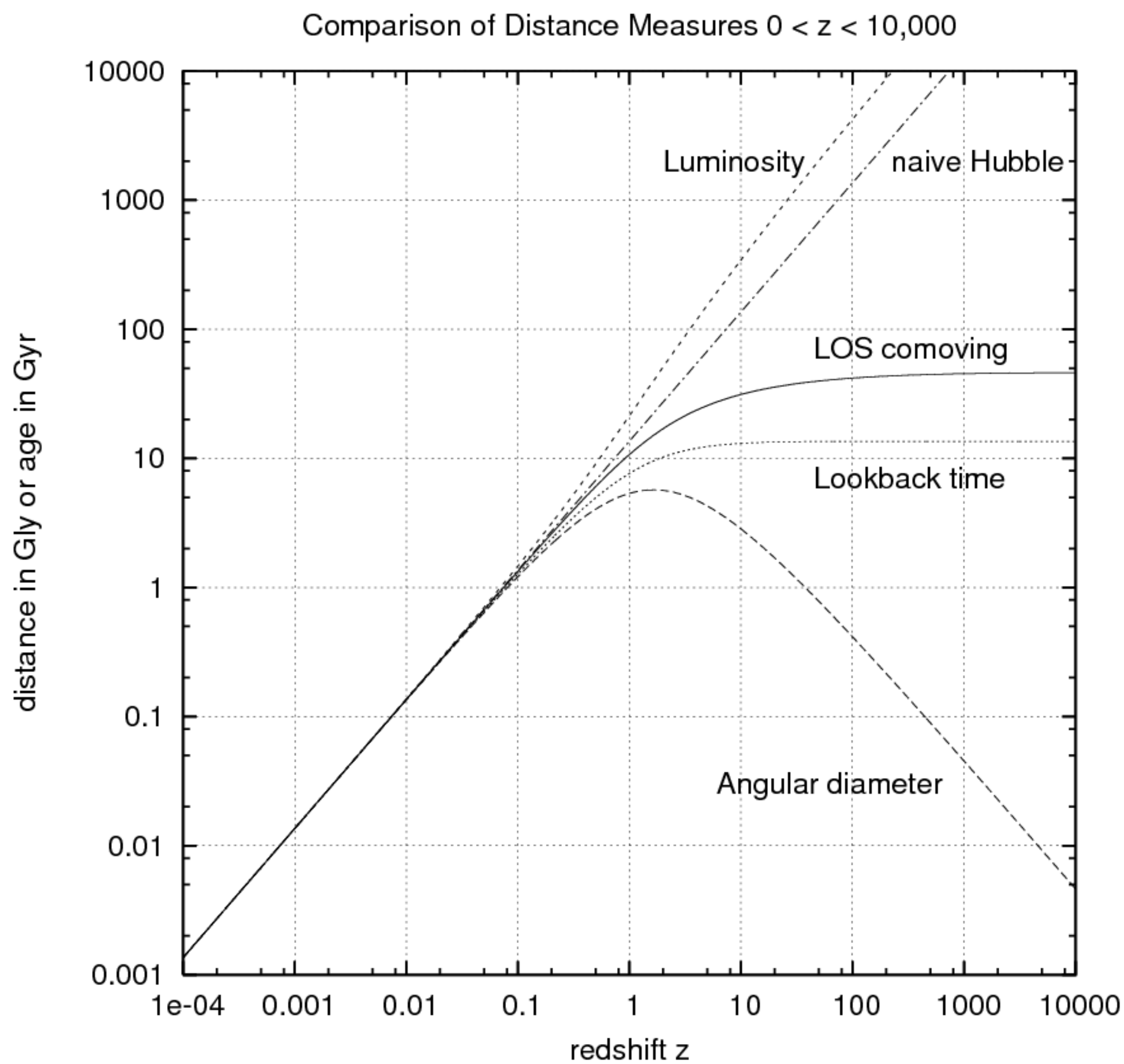

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$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

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- distances on the 2-sphere, embedded in $\mathbb{R}^3$

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$$
\begin{aligned}
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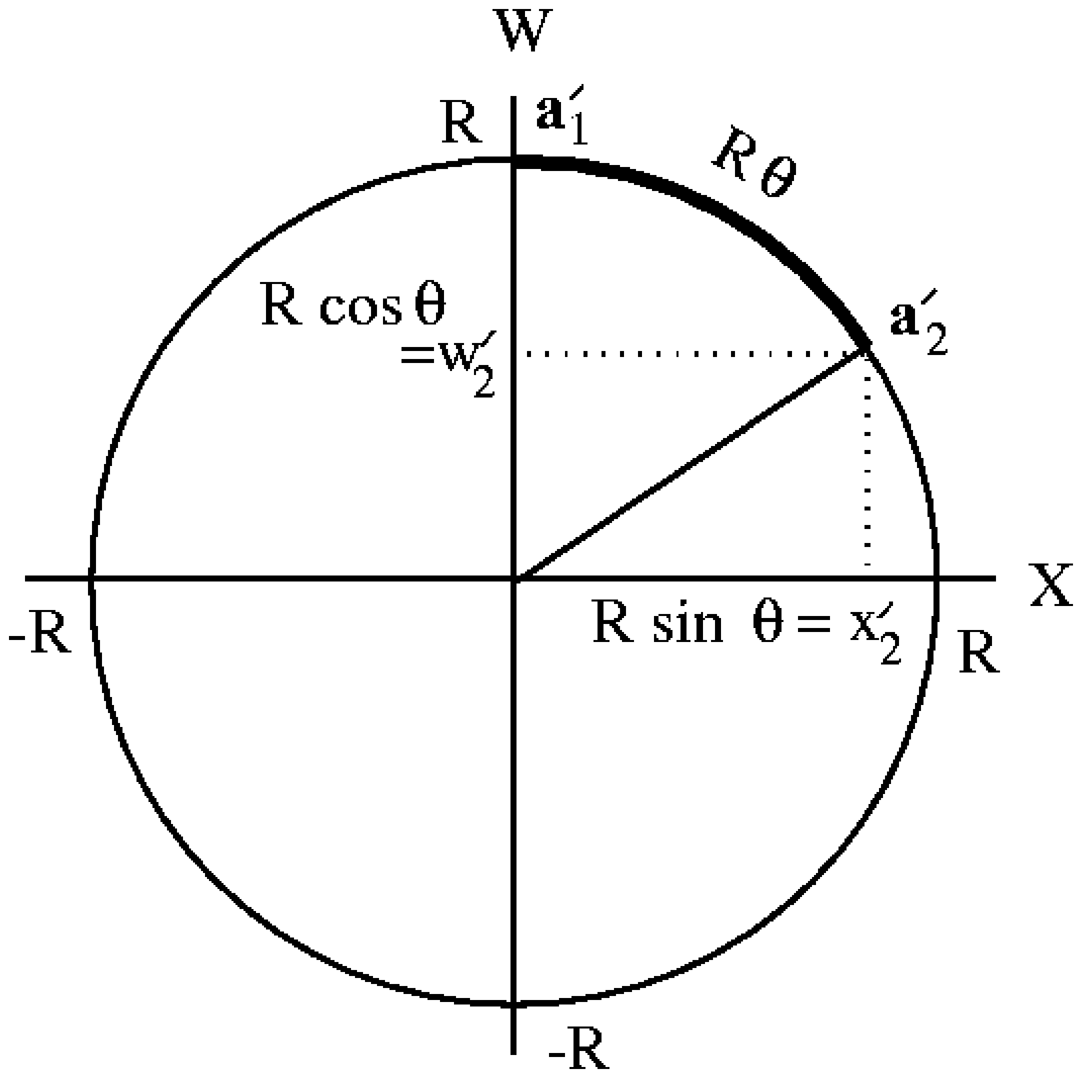

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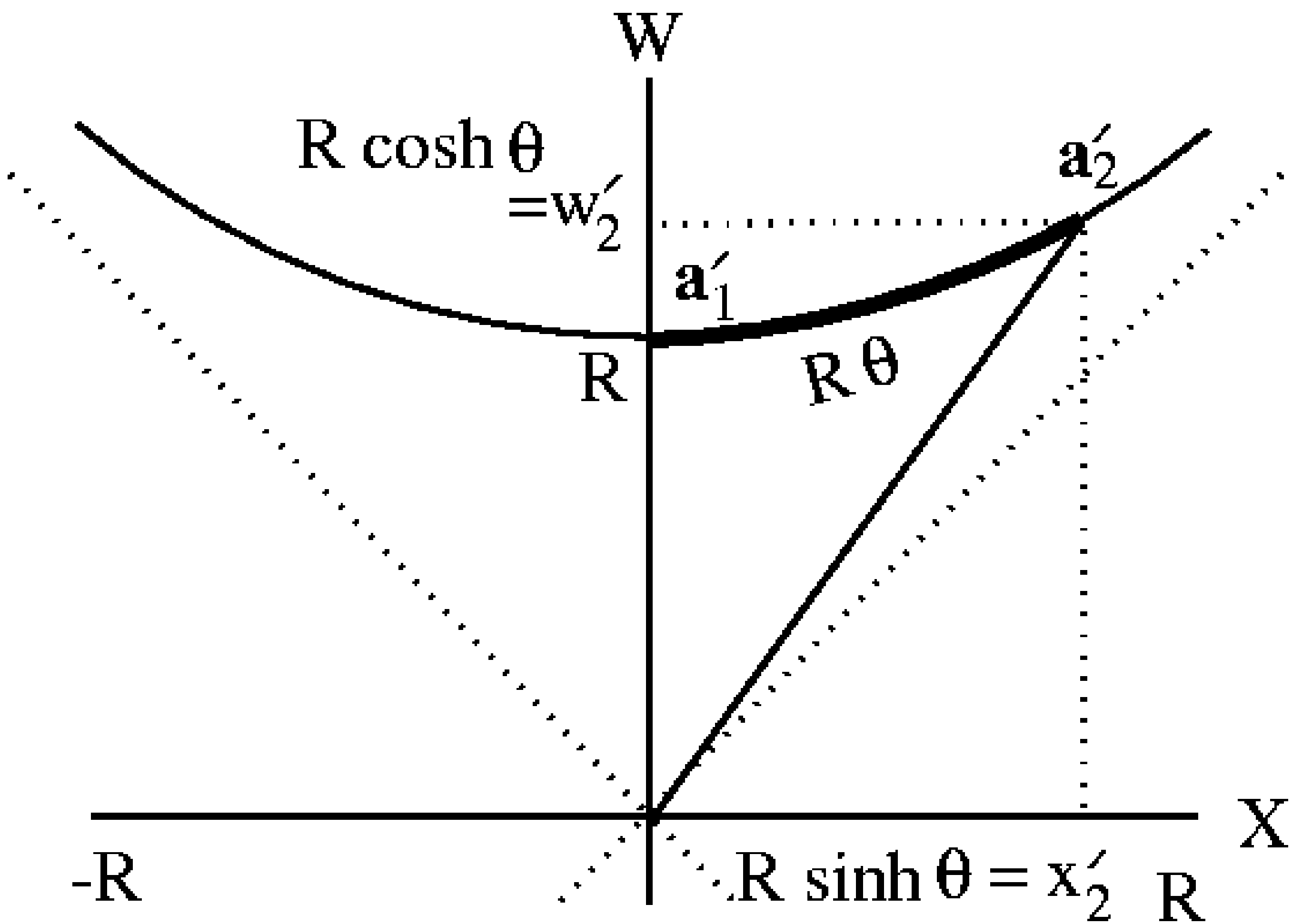

-

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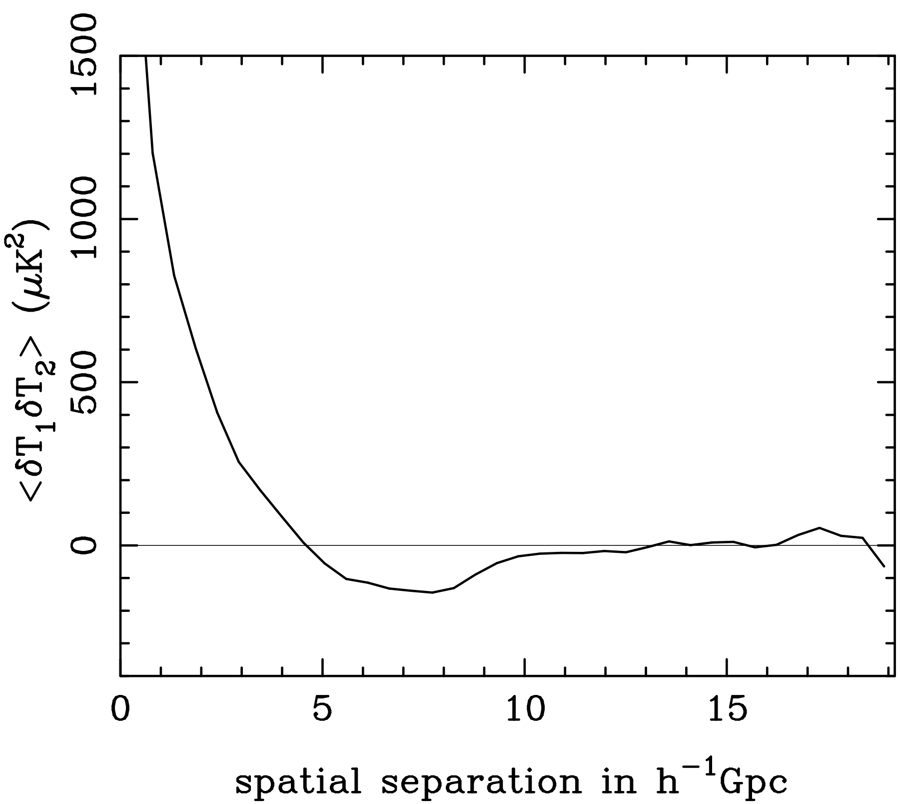

cf WMAP

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Linear perturbation theory

- Jan Ostrowski pdf 21.02.2013 (unpublished)