Monograph: Shape of the Universe

B.F. Roukema

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- verbal averaging: can we do better?

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- scalar (GR) averaging: statistically homogeneous spatial slices

verbal averaging

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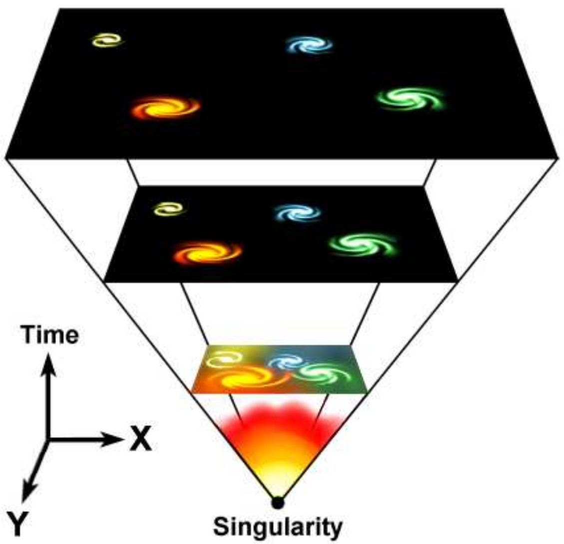
verbal averaging

- w:Comoving coordinates

verbal averaging



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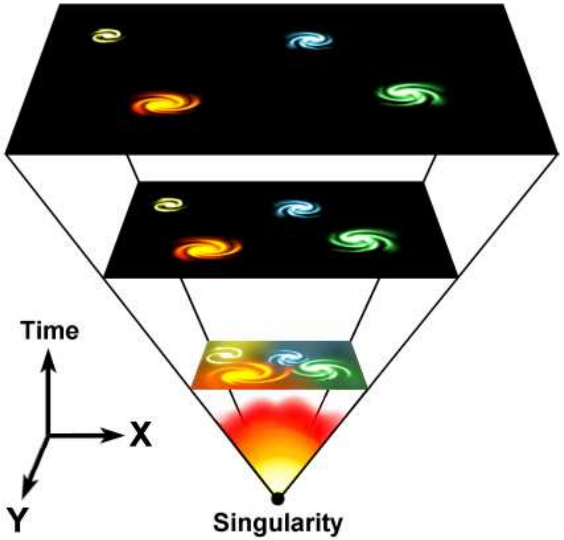

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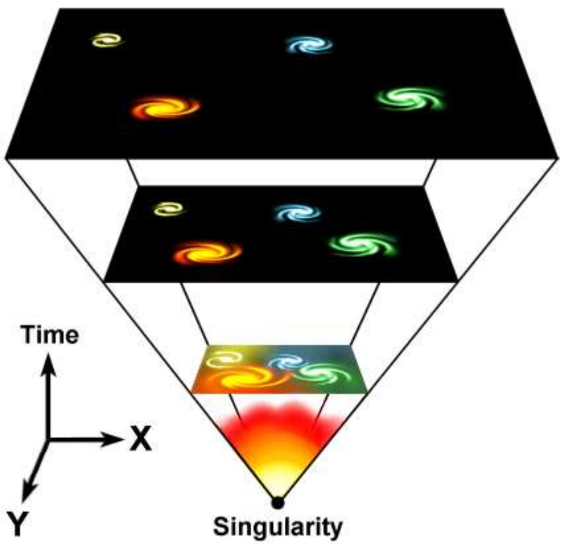
verbal averaging



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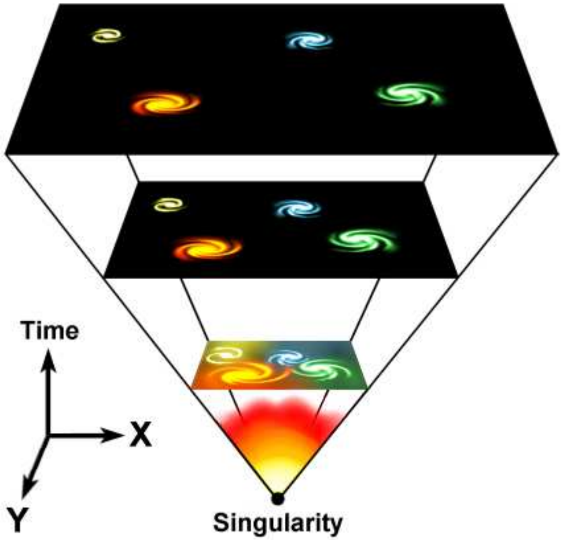
- spherical coordinates for spatial slice

verbal averaging



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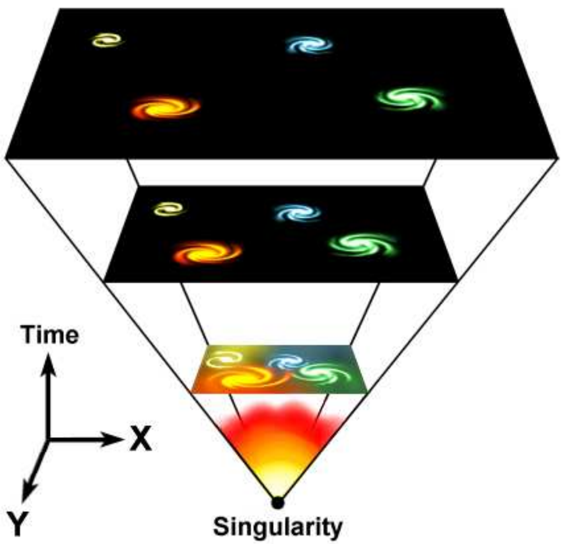

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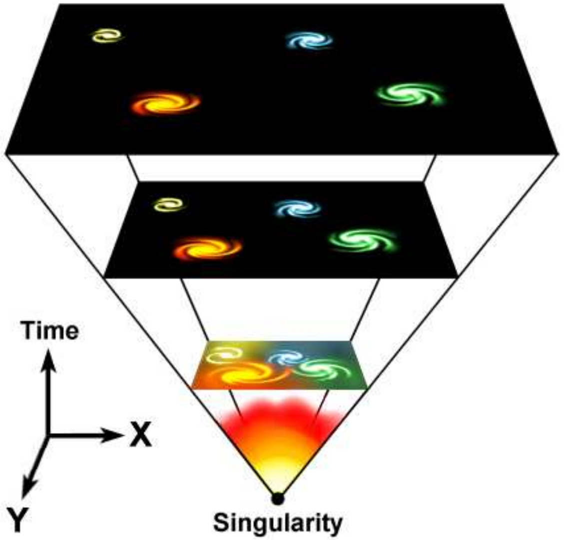

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- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

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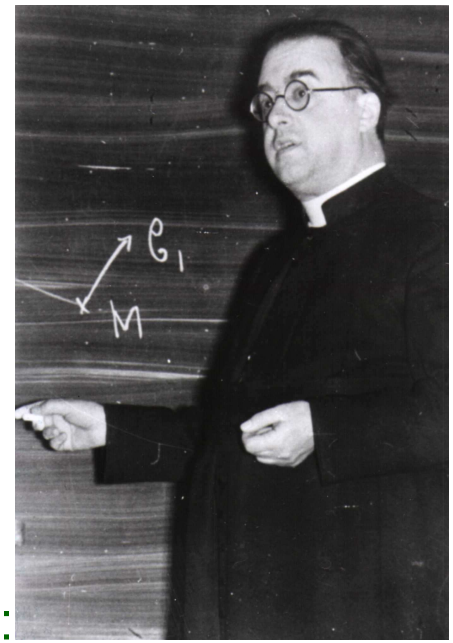

- w: w:
  w:Howard Percy Robertson
  w:Arthur Geoffrey Walker

FLRW metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \ldots$$

FLRW metric

$$ds^2 = -dt^2 + a^2(t)\,[\ldots]$$

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- but $\int \mathrm{d}u \neq$ proper time; *more:* arXiv:astro-ph/0707.2106

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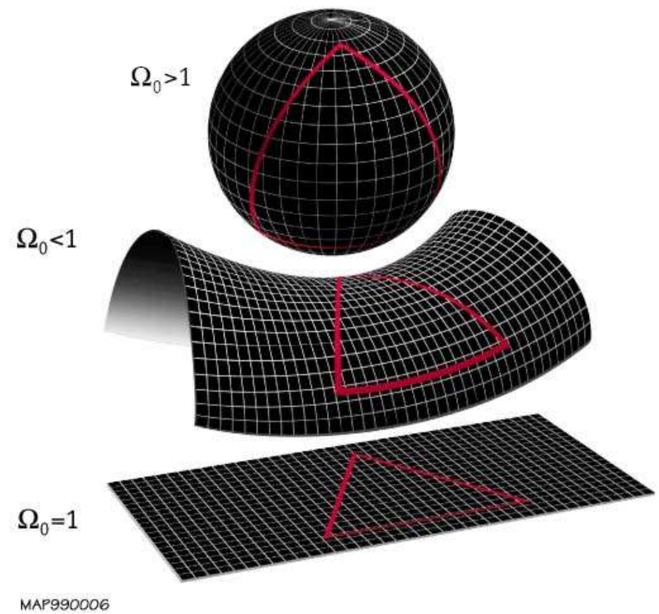
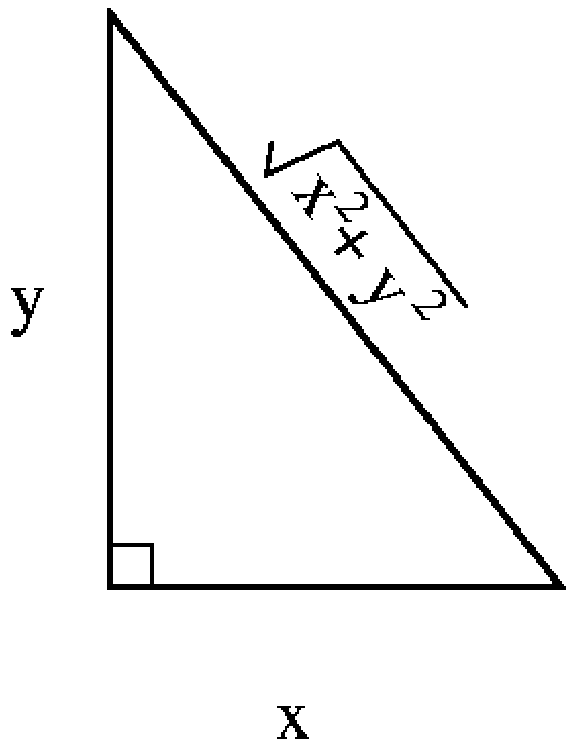


where $r_\perp := \begin{cases} R_\mathrm{C}\sinh\frac{r}{R_\mathrm{C}} & k<0 \\ r & k=0 \\ R_\mathrm{C}\sin\frac{r}{R_\mathrm{C}} & k>0 \end{cases}$

for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):



$$k = 0$$

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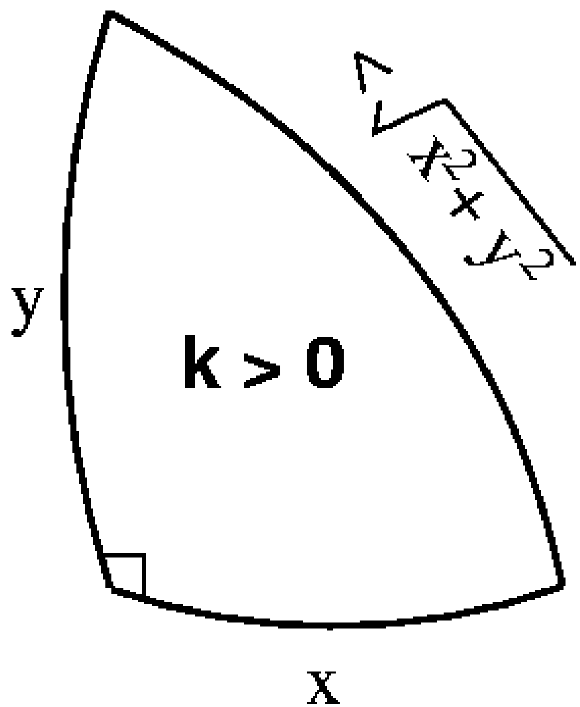

$$k > 0$$

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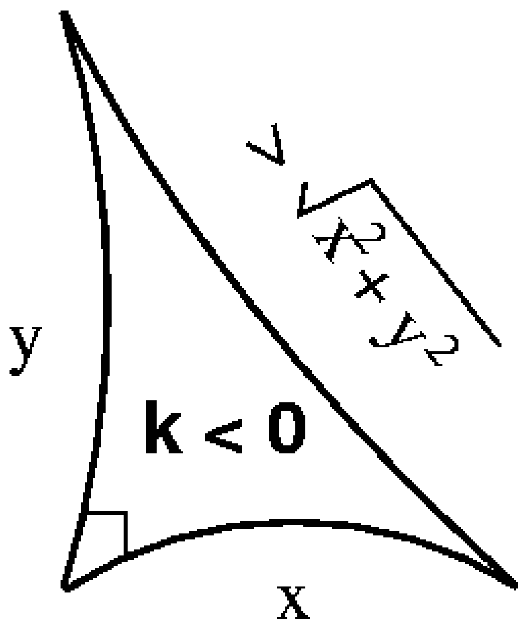

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2D curvature intuition: $k > 0$

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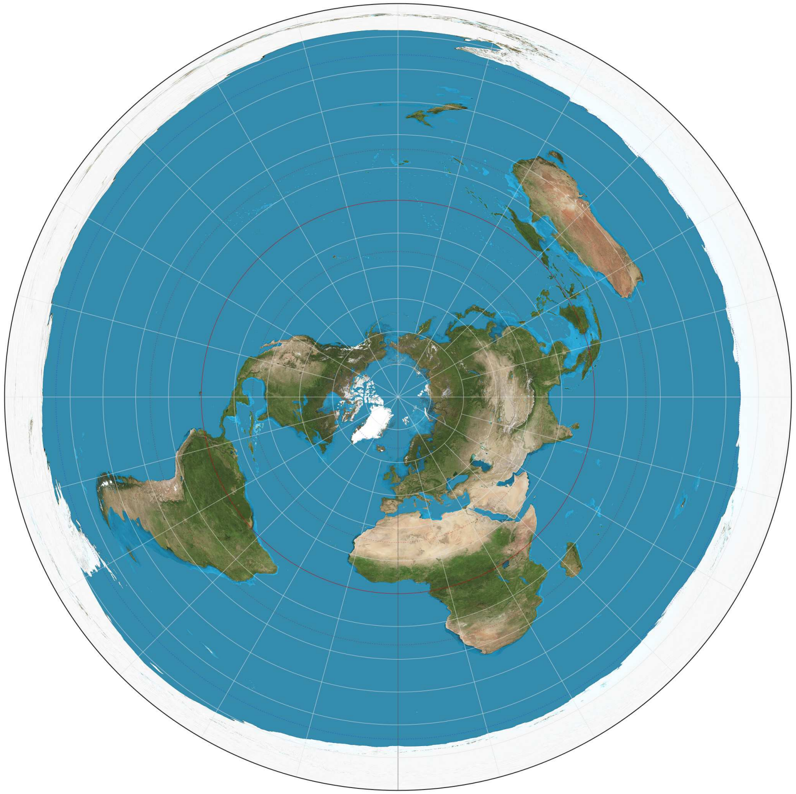

w: (al-Biruni, c. 1000 CE)

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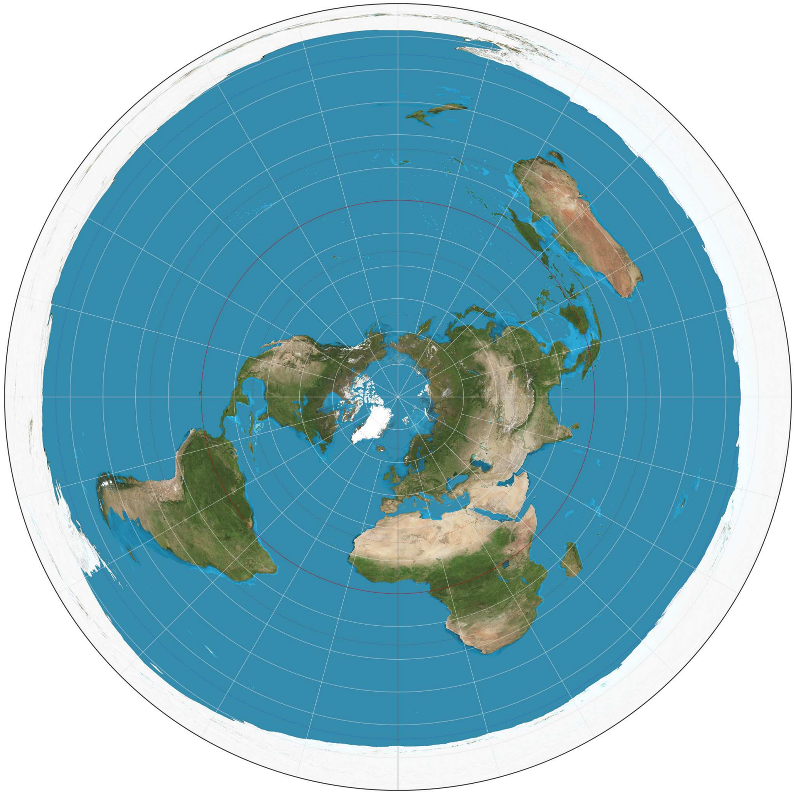

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- intuition switch: S^2 easier vs S^3 more physical

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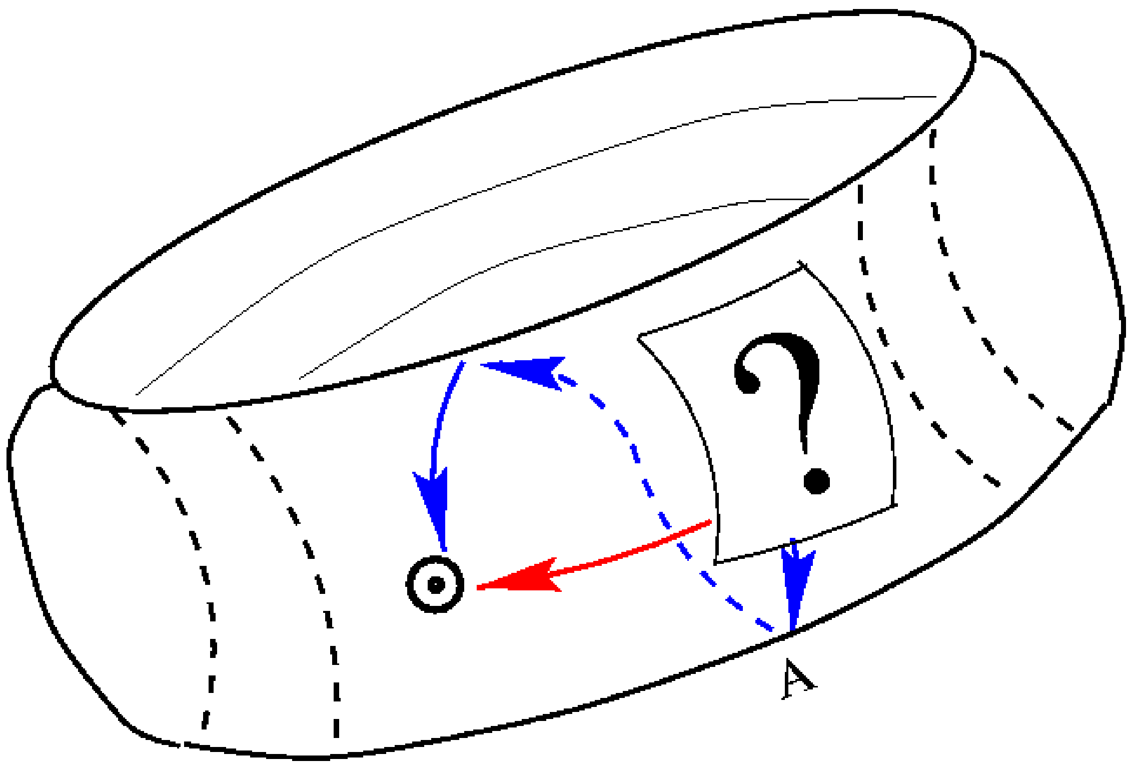

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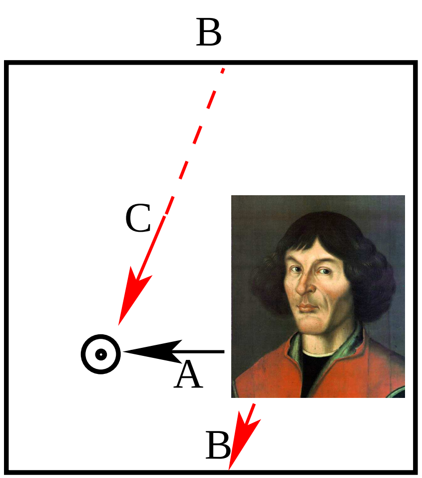

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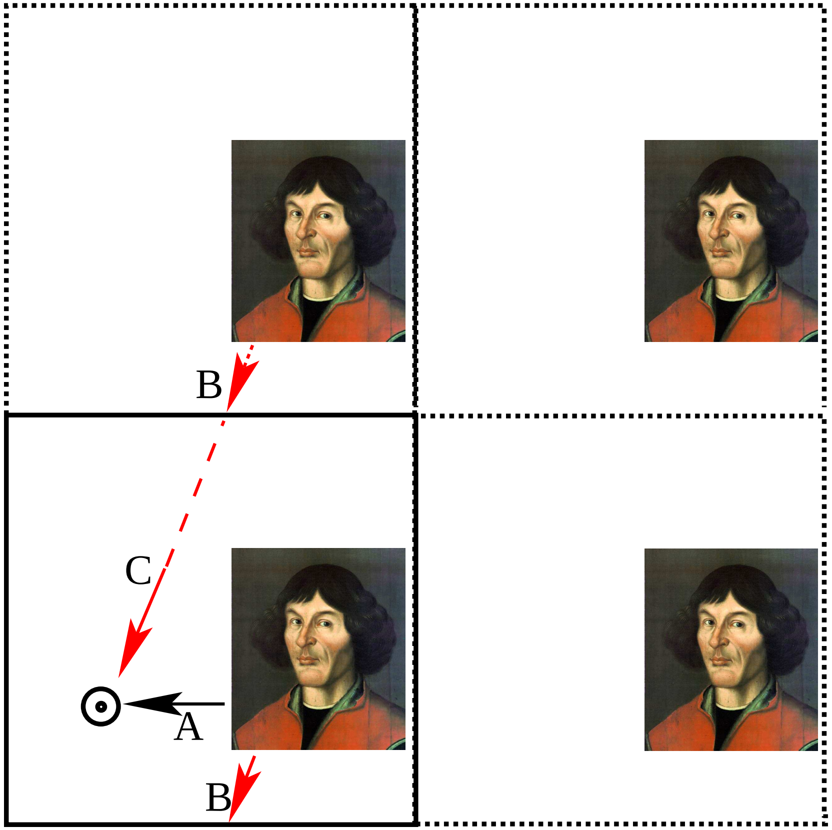

expansion

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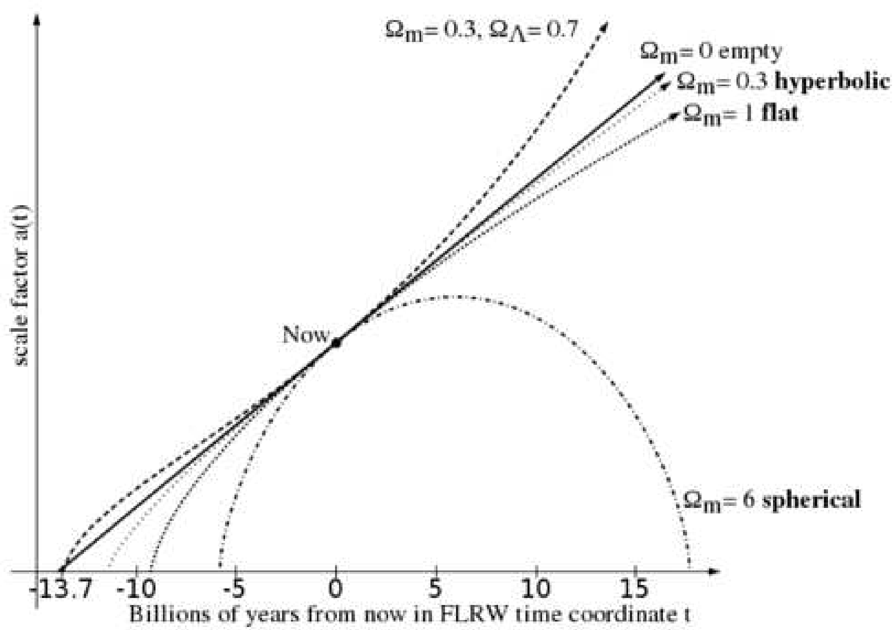

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  (Defn: $a_0 := 1$)

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  (Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_{\rm r} \propto a^{-4} = (1+z)^4$

Black body: COBE (~ 1992)

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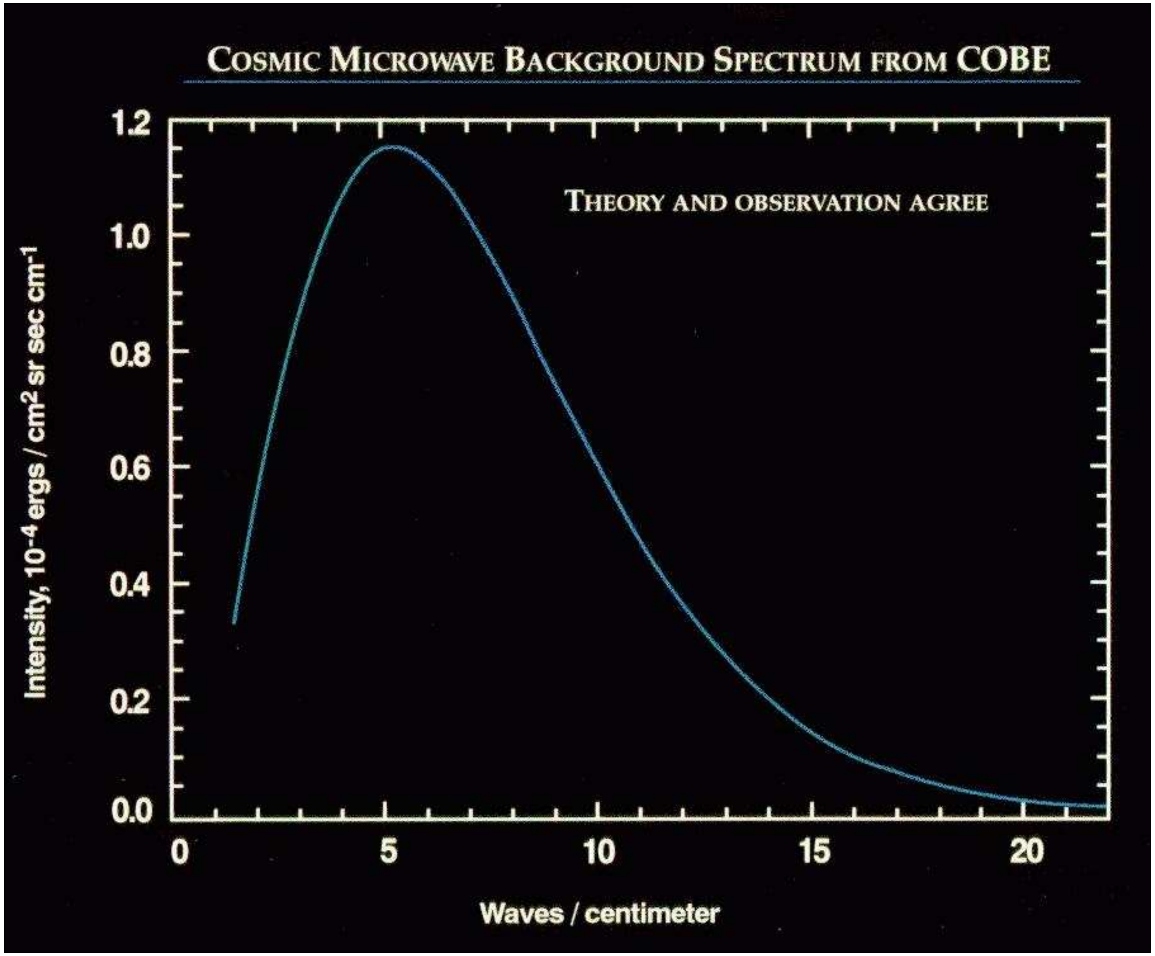

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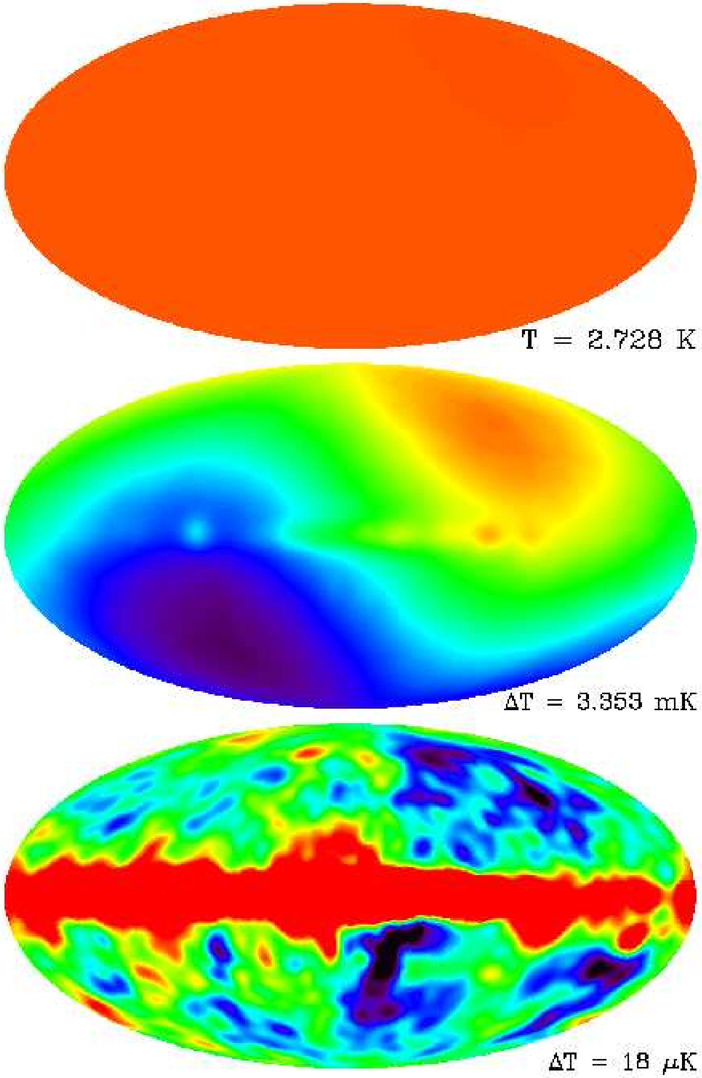
■ COBE /DMR (Differential Microwave Radiometer)



BBN: Big bang nucleosynthesis

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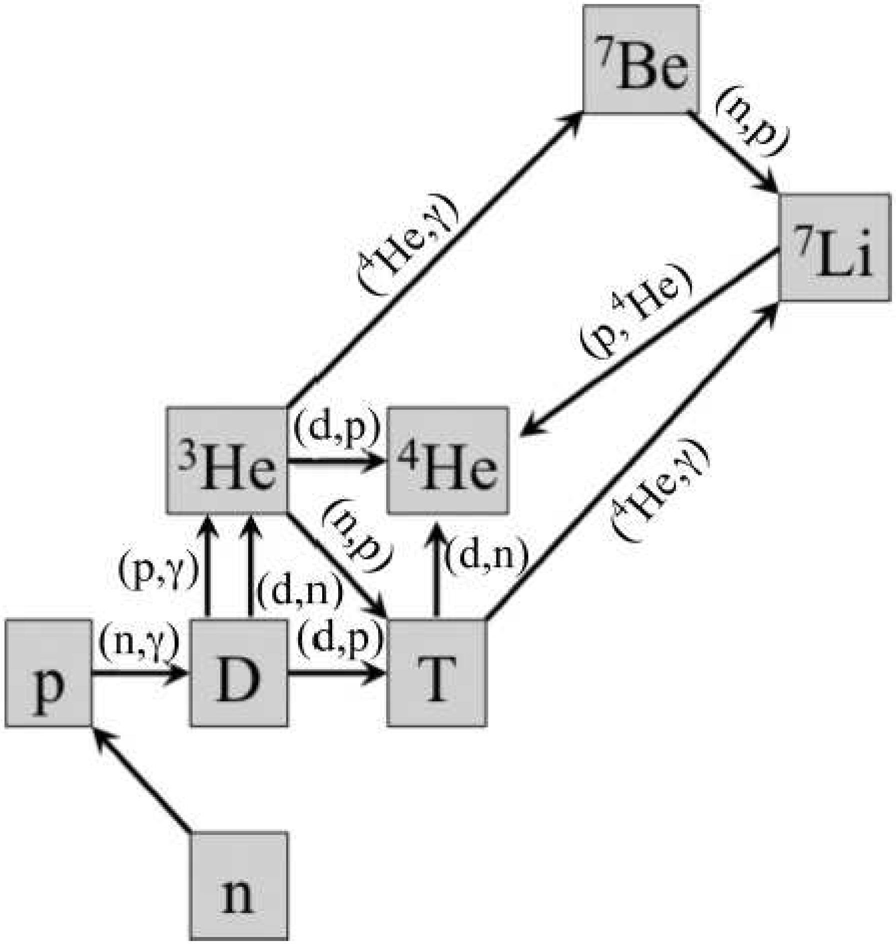

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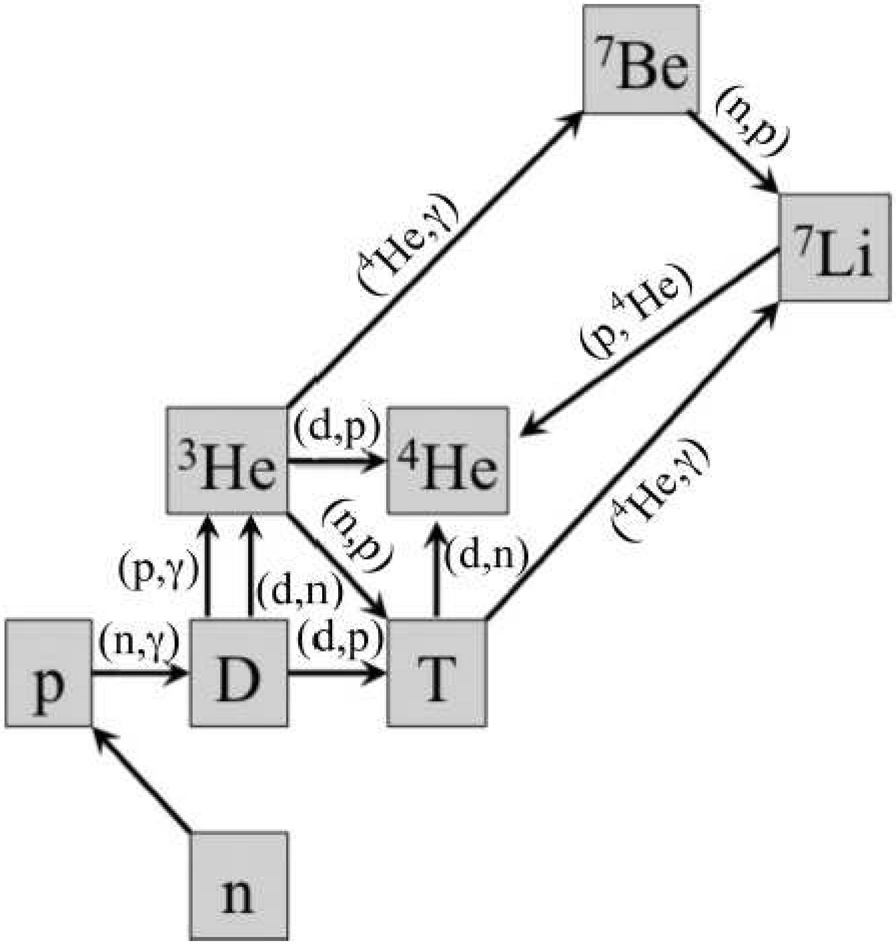

w:Big Bang nucleosynthesis

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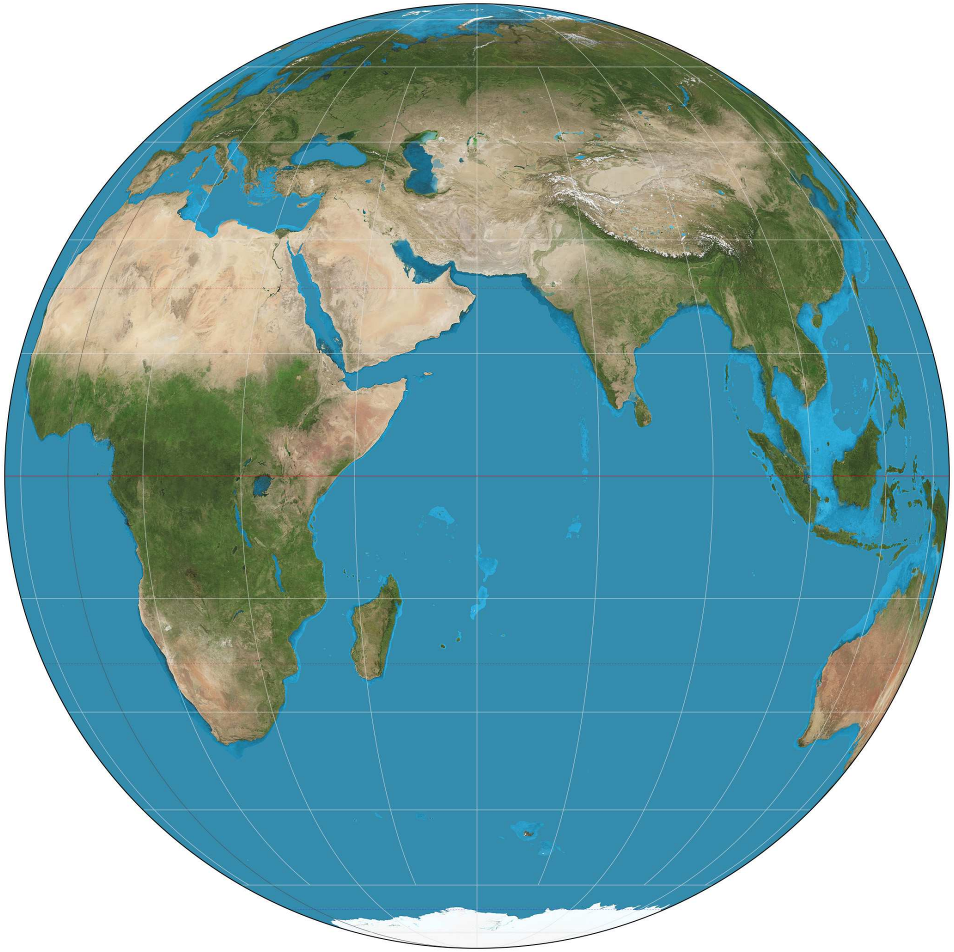

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- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- Friedmann Eqn: $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3}$$

FLRW: $a(t) =?$

- $$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \cos^2\theta d\phi^2)\right]$$
- universe content: $\mathrm{diag}(\mathbf{T}) = (-\rho, p, p, p)$
- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} = 8\pi\mathbf{T}$ and simplify: http://cosmo.torun.pl/Cosmo/FLRWEquationsGR
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- acceleration Eqn: $$\frac{\ddot{a}}{a} = -\frac{4\,\pi\,G\,(\rho + 3\,p/c^2)}{3}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $$\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$

- matter-dominated epoch: $\rho = \rho_{\mathrm{m}}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}}$

- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a^3} - \frac{c^2\,k}{a^2}}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$

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- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

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- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m}0}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m}0}\,a^{-3}$
- $k = 0$ case: $\dot{a} \propto a^{-1/2}$ for $a > 0$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$

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- $k = 0$ case: $\mathrm{d}a \propto a^{-1/2}\mathrm{d}t$ for $a > 0$

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- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$

- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

- $k = 0$ case: $a^{1/2}\mathrm{d}a \propto \mathrm{d}t$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $(2/3)a^{3/2} \propto t$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$

- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$

- $k = 0$ case: $a \propto t^{2/3}$

FLRW matter-dominated epoch

- Friedmann Eqn: $\boxed{\dot{a}^2 = \frac{8\,\pi\,G\,\rho_{\mathrm{m0}}}{3\,a} - c^2\,k}$
- matter-dominated epoch: $\rho = \rho_{\mathrm{m}} = \rho_{\mathrm{m0}}\,a^{-3}$
- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

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Defn: $\boxed{\text{Hubble parameter } H := \dot{a}/a}$

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$\Rightarrow$ Hubble constant $H_0 := H(z = 0)$

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$\Rightarrow$ $H(t) = \frac{2}{3t}$;

FLRW matter-dominated epoch

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$\Rightarrow$ $H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

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$\Rightarrow H(t) = \frac{2}{3t}$; $H_0 = H(t_0) = \frac{2}{3t_0}$

- convenient conversion: 1 km/s ≈ 1.04 kpc/Gyr ≈ 1 kpc/Gyr

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- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx 600$ km/s/Mpc

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- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx 0.6$ Gyr^{-1}

FLRW matter-dominated epoch

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- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

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- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
- Hubble (1929) ADS:1929PNAS...15..168H: $H_0 \approx$ 500 km/s/Mpc

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0}$

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

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- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

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- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

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- $k = 0$ case: $\boxed{a = \left(\frac{t}{t_0}\right)^{2/3}}$ Einstein–de Sitter model (EdS)

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- Lemaître (1927) ADS:1927ASSB...47...49L: $H_0 \approx$ 0.6 Gyr^{-1}
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$\Rightarrow$ EdS would give $t_0 = \frac{2}{3H_0} \approx 1.3$ Gyr $< t_{\mathrm{Earth}} \approx 4.5$ Gyr

- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1} $\Rightarrow t_0(\mathrm{EdS}) \approx 13.0$ or 6.5 Gyr, resp.

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$$

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\rho_{\rm m0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$ flat
  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

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- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$
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  - $\rho_{\rm m0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$ spherical
  - $\rho_{\rm m0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0$ hyperbolic

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$$

FLRW: ρ_{crit}

- Friedmann Eqn: $H^2 = \frac{8\,\pi\,G\,\rho}{3} - \frac{c^2\,k}{a^2}$

Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $$H^2 = \frac{\rho H^2}{\rho_{\rm crit}} - \frac{c^2\,k}{a^2}$$

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FLRW: ρ_{crit}

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Defn: $\boxed{\rho_{\mathrm{crit}} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\mathrm{m}} := \frac{\rho}{\rho_{\mathrm{crit}}}}$ matter density parameter

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 - \frac{c^2\, k}{a^2}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 - \frac{c^2\,k}{a^2}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2\,k}{a^2 H^2}}$ curvature density parameter

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 - \frac{c^2\,k}{a^2}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $H^2 = \Omega_{\rm m} H^2 + \Omega_k H^2$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

Defn: $\boxed{\rho_{\rm crit} := \frac{3H^2}{8\pi G}}$ critical density

Defn: $\boxed{\Omega_{\rm m} := \frac{\rho}{\rho_{\rm crit}}}$ matter density parameter

Defn: $\boxed{\Omega_k := -\frac{c^2 k}{a^2 H^2}}$ curvature density parameter (sign reversal!)

FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm m} + \Omega_k}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$

FLRW: ρ_{crit}

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  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
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FLRW: $\rho_{\rm crit}$

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FLRW: $\rho_{\rm crit}$

- Friedmann Eqn: $\boxed{1 = \Omega_{\rm tot} + \Omega_k}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic
- $\Omega_{\rm tot} :=$

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- $\Omega_{\rm tot} := \Omega_{\rm m} +$

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Defn: $\boxed{\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}}$ matter density parameter

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\text{m}0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\text{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\text{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\text{tot}} := \Omega_{\text{m}} + \Omega_{\text{r}} +$

FLRW: $\rho_{\rm crit}$

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- consider a fixed observation, e.g. $H_0 = 100$ km/s/Mpc
  - $\Omega_{\rm m0} = 1 \Leftrightarrow k = 0$ flat
  - $\Omega_{\rm m0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\rm m0} < 1 \Leftrightarrow k < 0$ hyperbolic
- $\Omega_{\rm tot} := \Omega_{\rm m} + \Omega_{\rm r} + \Omega_\Lambda$

FLRW: ρ_{crit}

- Friedmann Eqn: $\boxed{1 = \Omega_{\mathrm{tot}} + \Omega_k}$

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  - $\Omega_{\mathrm{m}0} > 1 \Leftrightarrow k > 0$ spherical
  - $\Omega_{\mathrm{m}0} < 1 \Leftrightarrow k < 0$ hyperbolic

- $\Omega_{\mathrm{tot}} := \Omega_{\mathrm{b}} + \Omega_{\mathrm{nbDM}} + \Omega_{\mathrm{r}} + \Omega_\Lambda$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$

FLRW curvature constant

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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: $R_{\rm C}$
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\rm C}^2 = 1 \Rightarrow k = 1/R_{\rm C}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; k = -\frac{\Omega_{k\,0} H_0^2}{c^2}$

FLRW curvature constant

- metric in
  - <u>azimuthal equidistant</u> coords: R_C
  - <u>orthographic</u> coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_\mathrm{C}^2 = 1 \Rightarrow k = 1/R_\mathrm{C}^2$
- <u>Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$</u> $\Rightarrow$ $R_\mathrm{C}^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \;\Rightarrow\; R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{\Omega_{k0}}$

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $R_{\mathrm{C}}^2 = -\frac{c^2}{H_0^2}\frac{1}{1-\Omega_{\mathrm{tot}\,0}}$

FLRW curvature constant

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- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
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- $\Omega_{\mathrm{tot}0} > 1$

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical*

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real

FLRW curvature constant

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat*

FLRW curvature constant

- metric in
  - azimuthal equidistant coords: R_{C}
  - orthographic coords: k
- orthographic: $1 - kr^2 = 0$ coord singularity at equator
- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
- Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow$ $\boxed{R_{\mathrm{C}} = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_{\mathrm{tot}0} - 1}}}$
- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined

FLRW curvature constant

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  - azimuthal equidistant coords: R_{C}
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- $\Rightarrow kR_{\mathrm{C}}^2 = 1 \Rightarrow k = 1/R_{\mathrm{C}}^2$
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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic*

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- $\Omega_{\mathrm{tot}0} > 1$ *spherical* R_{C} real
- $\Omega_{\mathrm{tot}0} = 1$ *flat* R_{C} undefined
- $\Omega_{\mathrm{tot}0} < 1$ *hyperbolic* R_{C} imaginary (or use $|R_{\mathrm{C}}|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR

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- MAXIMA: calculate $\mathbf{G}$ and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
  http://cosmo.torun.pl/Cosmo/FLRWEquationsGR
- *hint:* mixed index form of $\mathbf{g}$ is easy

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

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Defn: "dust solution": $p(t) = 0$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2}{2}\frac{\rho}{\rho_{\mathrm{crit}}} + \Omega_\Lambda\,H^2$$

Einstein's free parameter: Λ

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Friedmann Eqn ($\Lambda \neq 0$): $$\frac{c^2\, k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\, G\, \rho}{3} + \frac{c^2\, \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a} = -\frac{H^2\,\Omega_\mathrm{m}}{2} + \Omega_\Lambda\, H^2$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

acceleration Eqn ($\Lambda \neq 0$): $$\frac{\ddot{a}}{a}\frac{a^2}{\dot{a}^2} = -\frac{\Omega_\mathrm{m}}{2} + \Omega_\Lambda$$

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Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

acceleration Eqn ($\Lambda \neq 0$): $$q = \frac{\Omega_{\mathrm{m}}}{2} - \Omega_\Lambda$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2\,\Lambda}{3\,H^2}$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ "deceleration parameter"

- $\boxed{q = \frac{\Omega_{\rm m}}{2} - \Omega_\Lambda}$

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- $\boxed{q = \frac{\Omega_\mathrm{m}}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Lambda = 0$ and $\Omega_\mathrm{m} > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
  ADS:1917SPAW.......142E

Friedmann Eqn ($\Lambda \neq 0$):
$$\frac{c^2\,k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G\,\rho}{3} + \frac{c^2\,\Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

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distances in FLRW cosmology

- azimuthal equidistant r:

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$\Rightarrow$ $\dot{a}^2 = H_0^2 \left(\Omega_{\mathrm{m}0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2\right)$

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Low z limit Hubble law

- use $\underline{r = \text{comoving radial distance}}$

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$\Rightarrow \frac{\mathrm{d}r}{\mathrm{d}z} \approx \frac{c}{H_0}$

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$\Rightarrow$ when $0 < z \ll 1$, $\beta := v/c$, $r \approx \dfrac{z\,c}{H_0}$

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&\approx -\int_z^0 \frac{c}{H_0}\frac{1\,\mathrm{d}z}{\sqrt{1}} \\
&= -\int_z^0 \frac{c}{H_0}\,\mathrm{d}z
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EdS: radial comoving distance

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&= \frac{c}{H_0} \left[2a^{1/2}\right]_{1/(1+z)}^{1} \\
&= \boxed{\frac{2c}{H_0}\left(1 - \frac{1}{\sqrt{1+z}}\right)} \\
&\approx 6h^{-1}\mathrm{Gpc}\left(1 - \frac{1}{\sqrt{1+z}}\right)
\end{aligned}$$

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- high-level frontends (e.g. python) should be easy to write

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- $d_{\mathrm{A}} = r_\perp\, a$ (scaled by the scale factor)

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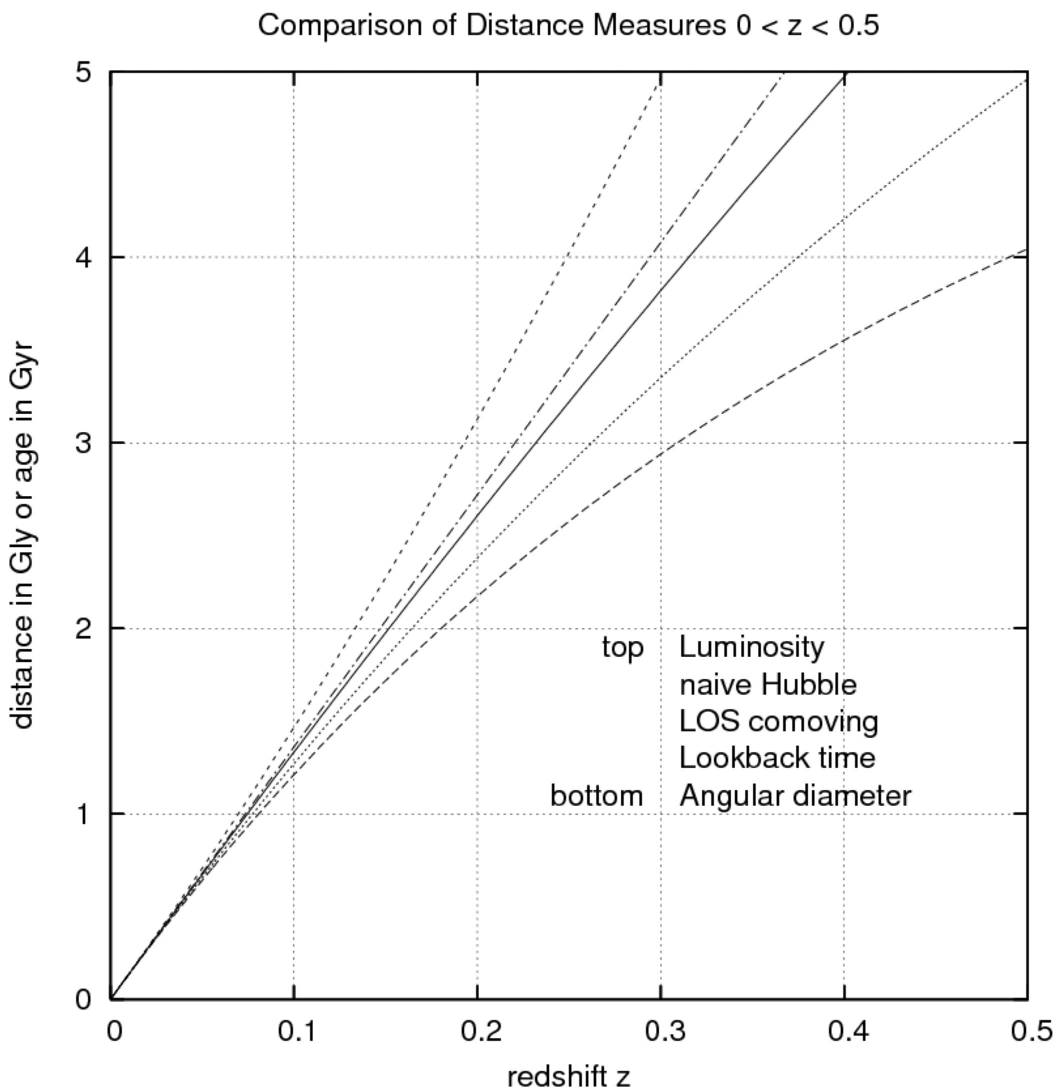
- w:Distance measures (cosmology)

FLRW distances e.g. ΛCDM



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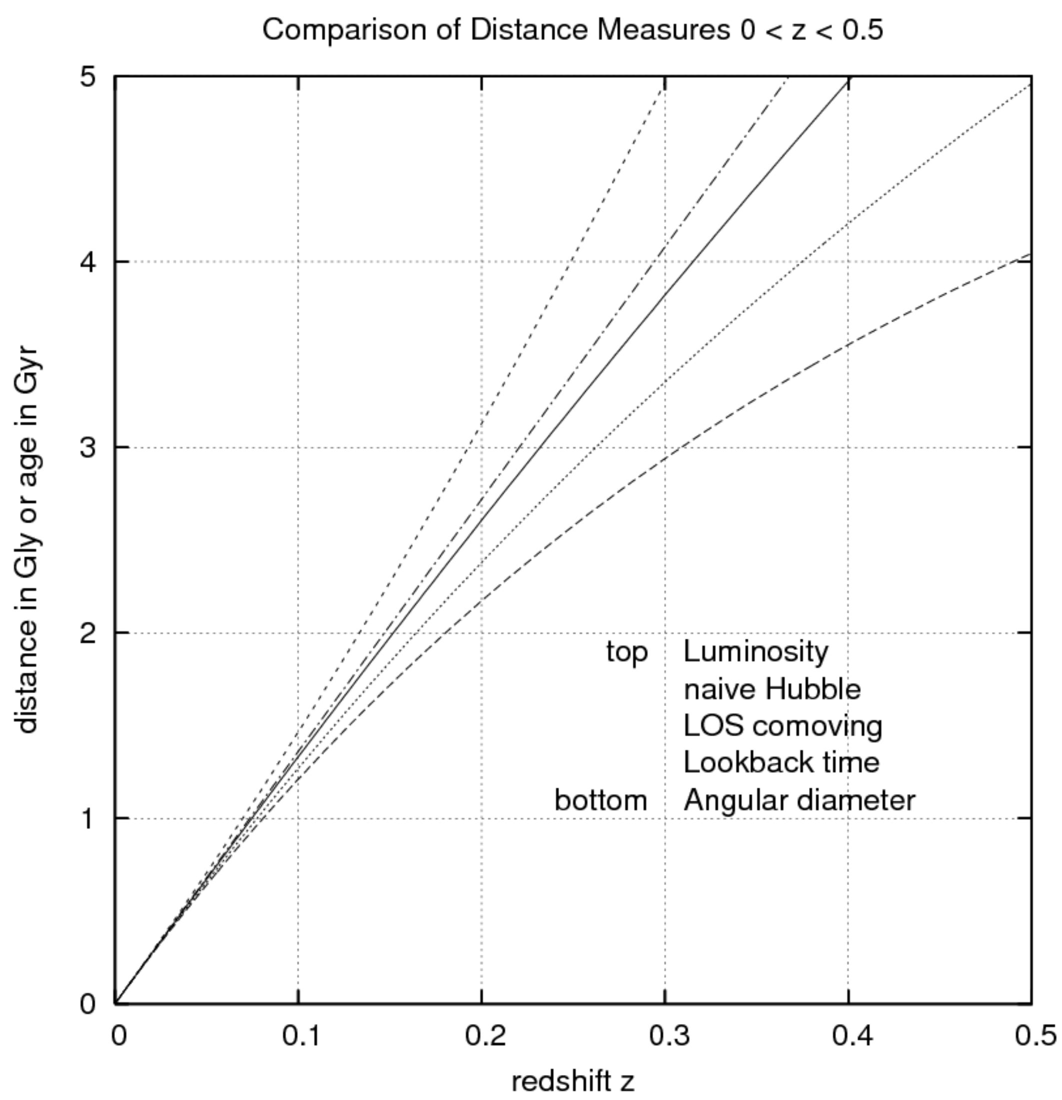

Defn: $\boxed{h := H_0/100 \text{ km/s/Mpc}}$ (without a "0" subscript on h)

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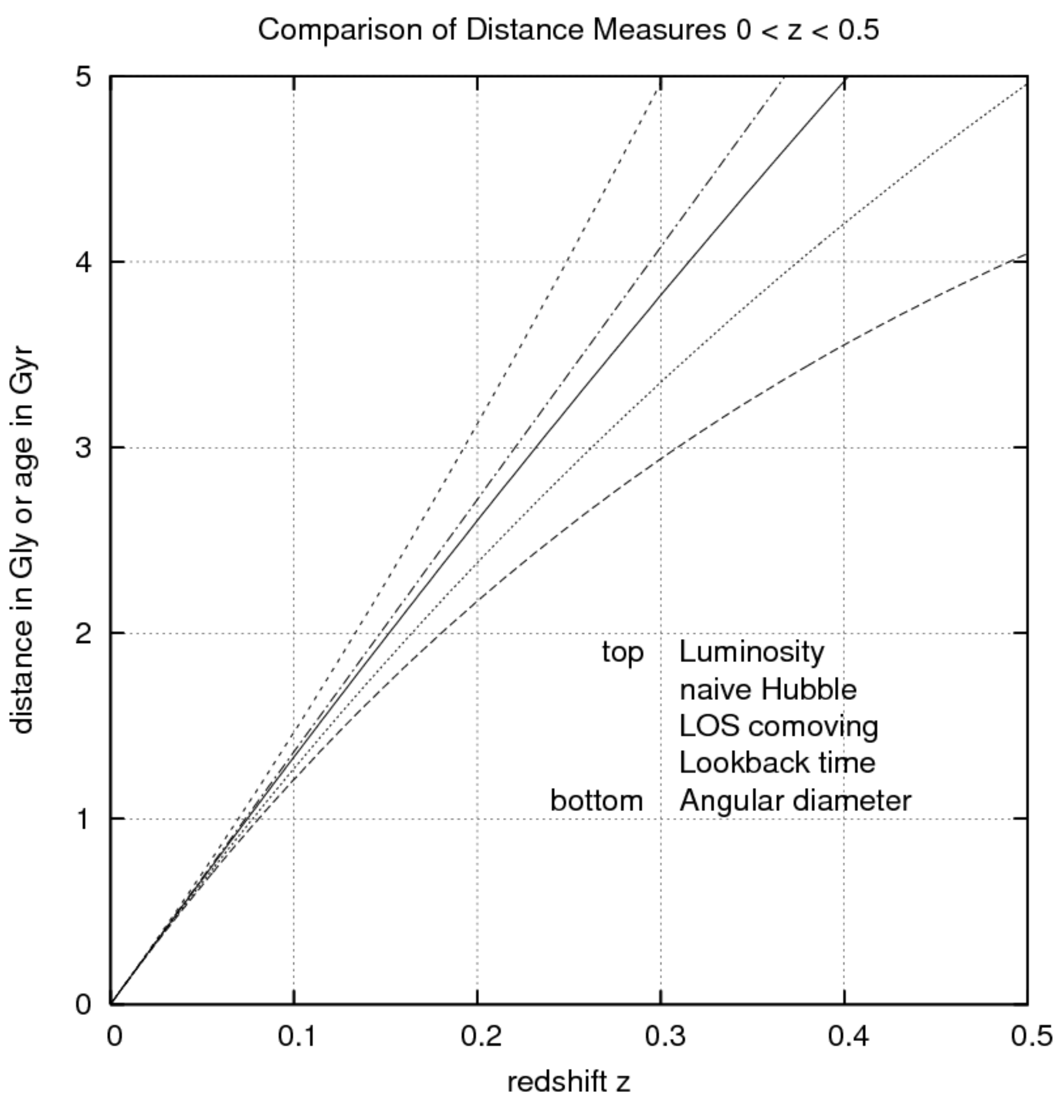

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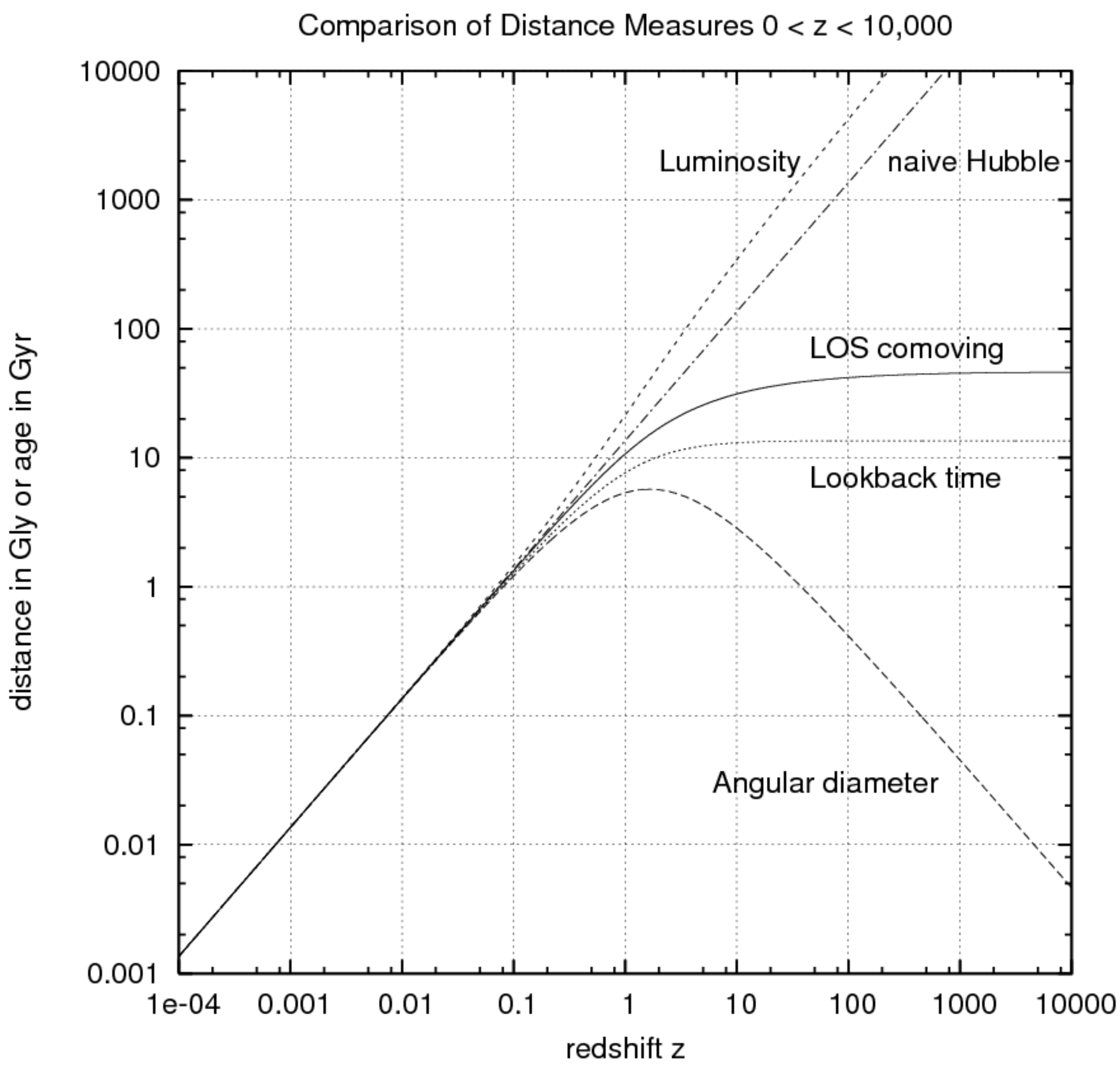

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  3. $v_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\mathrm{d}\left(\int_0^r \mathrm{d}s|_{t,\theta,\phi}\right)}{\mathrm{d}t} = \frac{\mathrm{d}(ar)}{\mathrm{d}t} = r\,\dot{a} = r H_0$ or
  4. $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}} := \frac{\int_0^r \mathrm{d}s|_{t,\theta,\phi}}{t_0} = \frac{ar}{t_0} = \frac{r}{t_0}$
- $v_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq c/H_0 \approx 3\mathrm{h}^{-1}$ Gpc
- $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}} \geq c$ when $r \geq ct_0 \approx 4.5\mathrm{h}^{-1}$ Gpc
- $v_{\mathrm{gal}}^{\mathrm{glob}}$, $\langle v\rangle_{\mathrm{gal}}^{\mathrm{glob}}$ = theoretical, global $\neq$ velocity in tangent spacetime

$\Rightarrow$ no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature $(+, 0, -)$?

Non-radial spatial geodesics

- distances on the 2-sphere, embedded in $\mathbb{R}^3$

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Non-radial spatial geodesics

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Non-radial spatial geodesics

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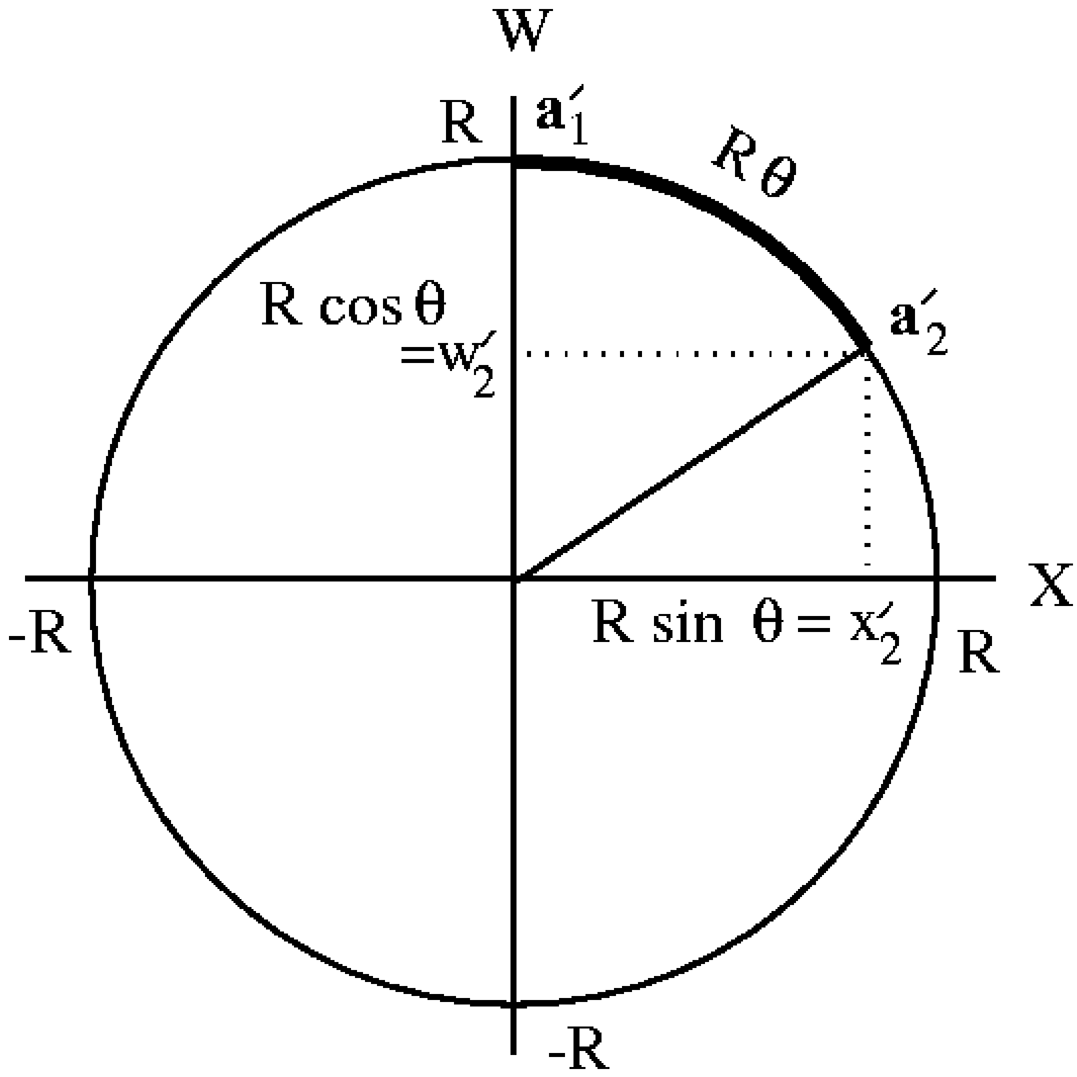

Non-radial spatial geodesics

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Non-radial spatial geodesics

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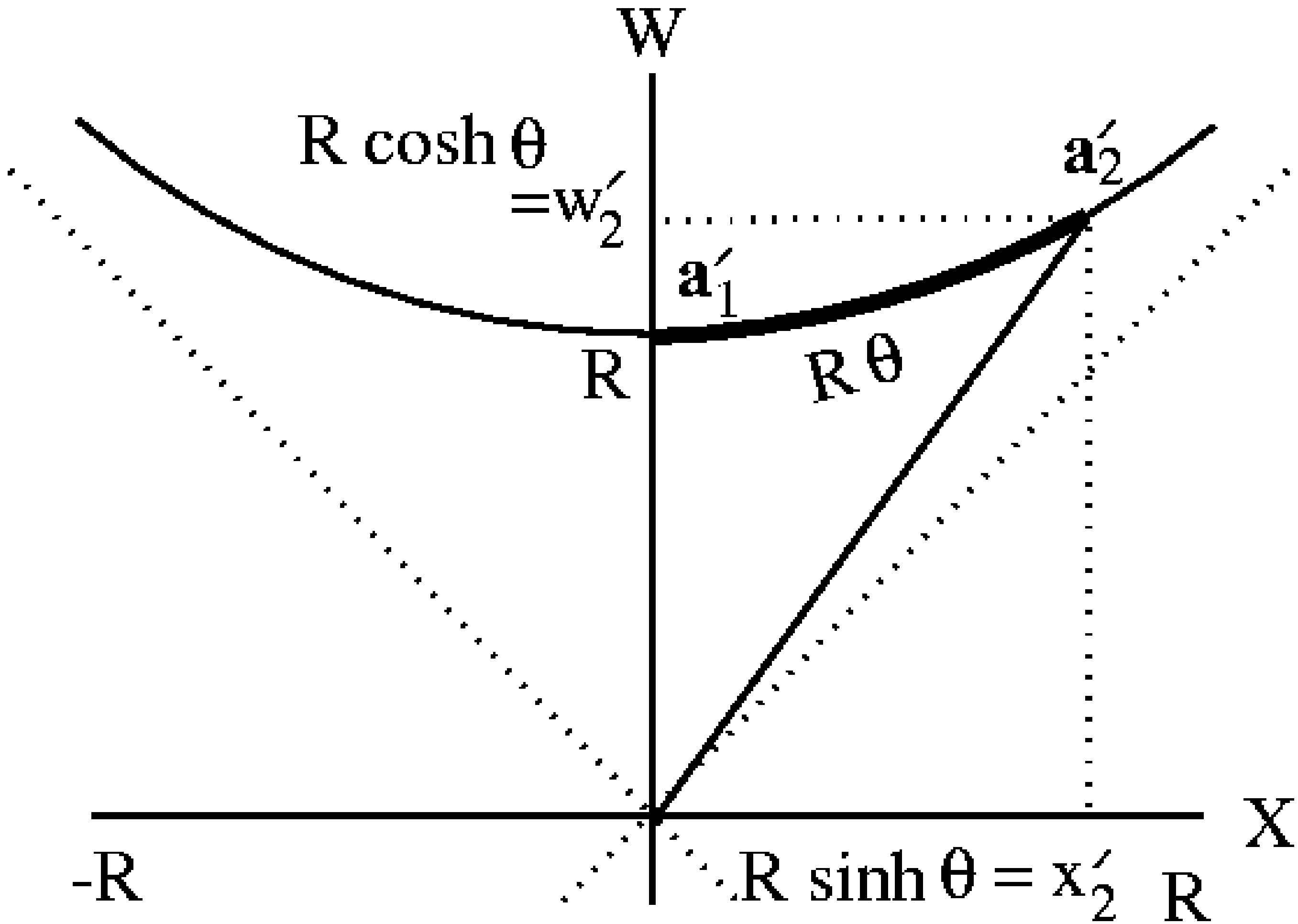

-

distances on $\mathsf{S}^3 \subset \mathbb{R}^4$ or $\mathsf{H}^3 \subset \mathsf{M}^4$

- $$\Sigma(\chi_i) := \begin{cases} R_\mathrm{C}\ \sinh(\chi_i / R_\mathrm{C}) & k < 0 \\ \chi_i & k = 0 \\ R_\mathrm{C}\ \sin(\chi_i / R_\mathrm{C}) & k > 0 \end{cases}$$

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- metric on S^3 (or $\mathbb{R}^3$ or H^3):
$$\mathrm{d}s^2 = \begin{cases} (k/|k|)\ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \mathrm{d}w^2 & k \neq 0 \\ \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 & k = 0 \end{cases}$$

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- inner product:
$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \equiv \begin{cases} (k/|k|) \ (x_1 x_2 + y_1 y_2 + z_1 z_2) + w_1 w_2 & k \neq 0 \\ x_1 x_2 + y_1 y_2 + z_1 z_2 & k = 0 \end{cases}$$

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