



Monograph: Shape of the Universe

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24 April 2014



Introduction

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◆ verbal averaging: can we do better?

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- ◆ but: \exists galaxies \Rightarrow inhomogeneous, anisotropic spatial slices

- ◆ standard model: **density perturbations (anisotropy)**

- ◆ scalar (GR) averaging: statistically homogeneous spatial slices

■ within this model, what is the shape of the Universe?

verbal averaging

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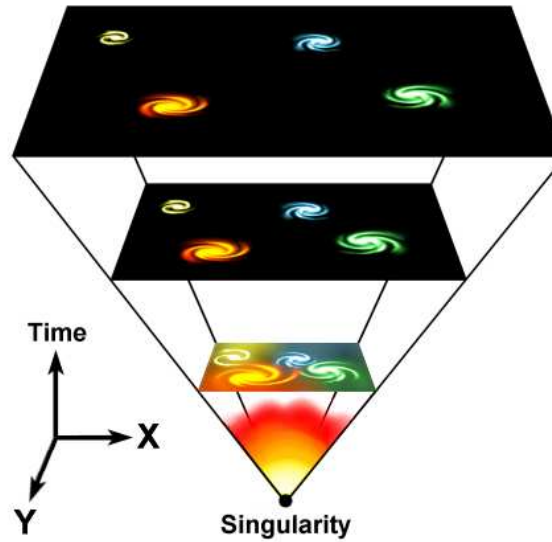
- practical meaning:

1. assume homogeneity and isotropy
2. find the (differential 4-pseudo-manifold, metric) pairs (M, \mathbf{g}) that solve $\mathbf{G} = 8\pi\mathbf{T}$
3. assume that (M, \mathbf{g}) remains unchanged if we add density perturbations to an early time slice

verbal averaging

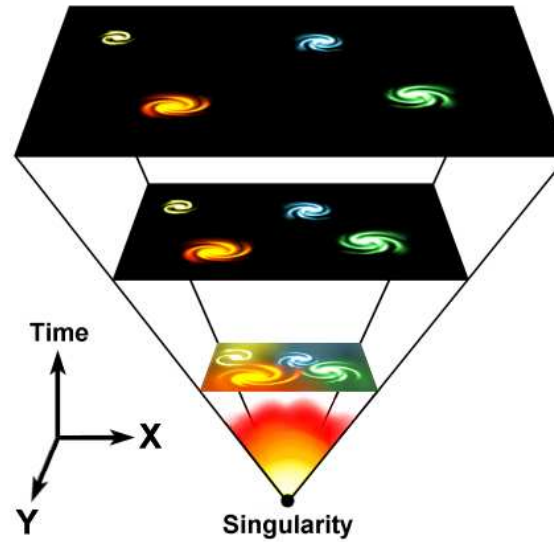
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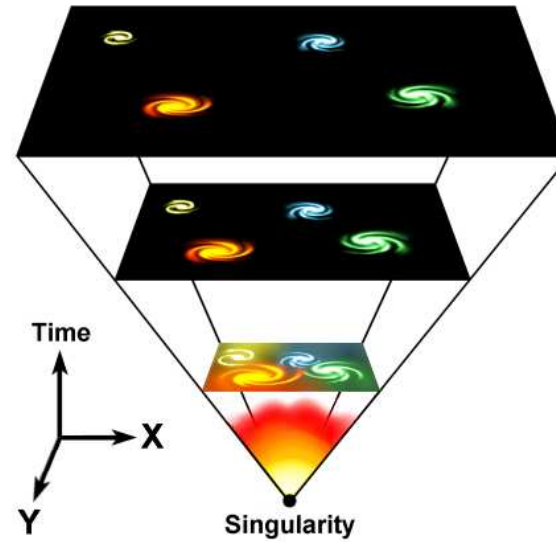


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$$\Delta x(t) = a(t) \Delta r$$

verbal averaging



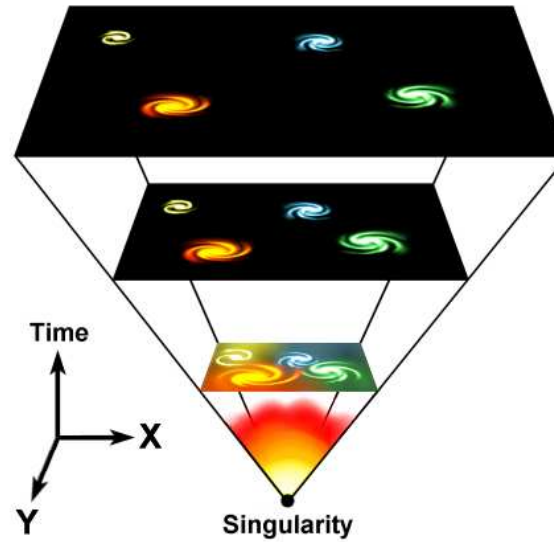
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- spherical coordinates for spatial slice

verbal averaging

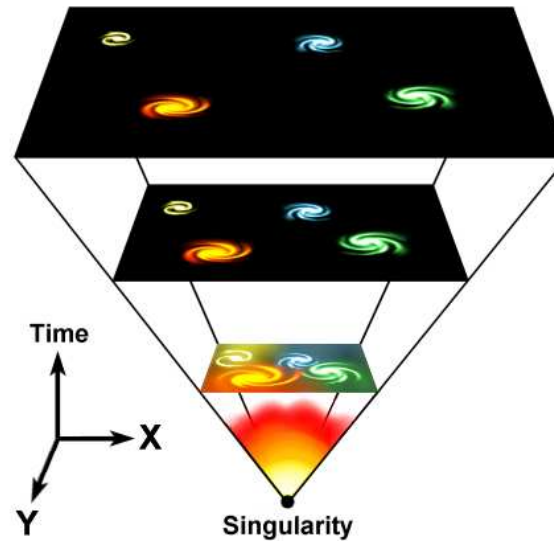


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$$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} ds = a(t)\Delta r = a(t)|r_2 - r_1|$$

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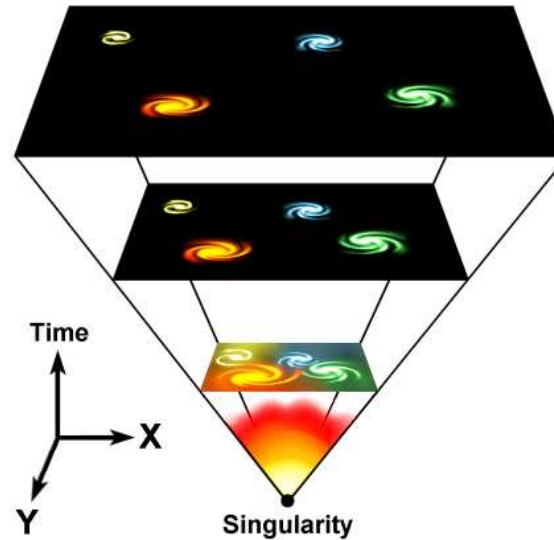
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$$\int_{(t,r_1,\theta,\phi)}^{(t,r_2,\theta,\phi)} ds = a(t)\Delta r = a(t)|r_2 - r_1|$$

where all expansion/contraction \rightarrow w: scale factor $a(t)$

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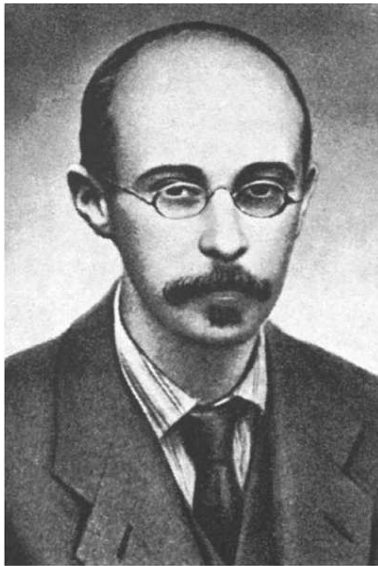
- universe is static in comoving coordinates (r, θ, ϕ)

FLRW metric

- w:Friedmann–Lemaître–Robertson–Walker metric

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- w: *A.P. Robertson* w:

w:Howard Percy Robertson

w:Arthur Geoffrey Walker

FLRW metric

$$ds^2 = -dt^2 + \dots$$

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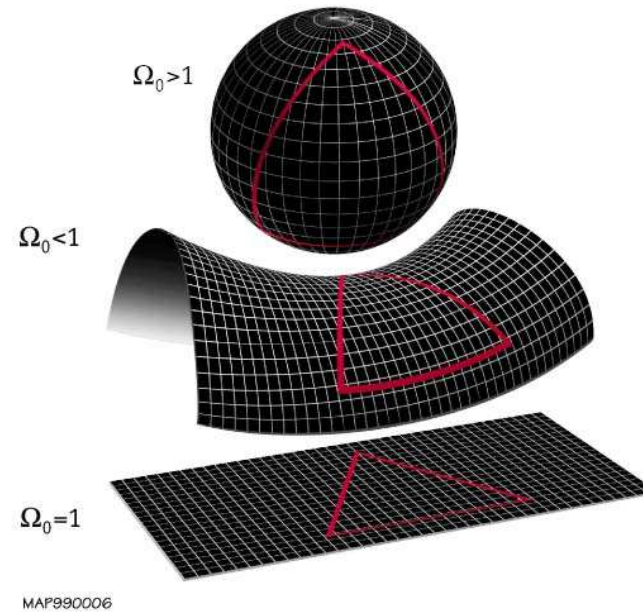
FLRW metric

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$$\text{where } r_{\perp} := \begin{cases} R_C \sinh \frac{r}{R_C} & k < 0 \\ r & k = 0 \\ R_C \sin \frac{r}{R_C} & k > 0 \end{cases}$$

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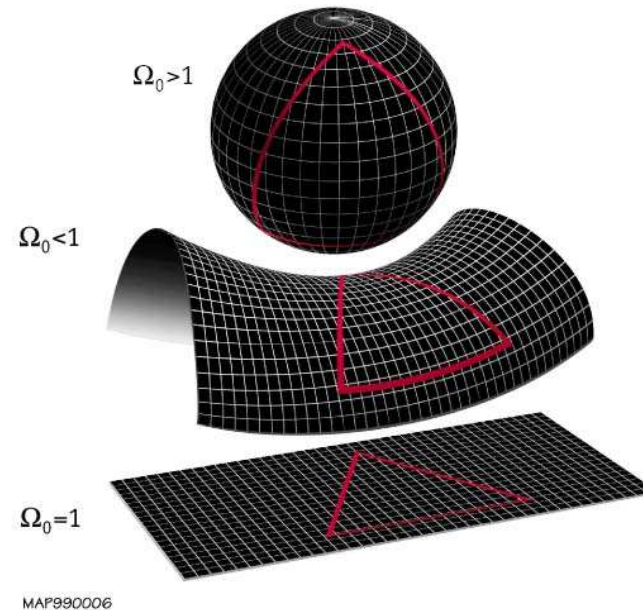


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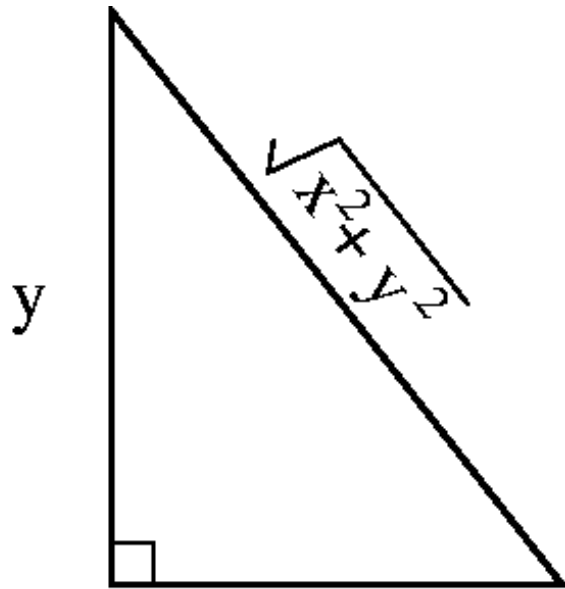
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for a comoving radius of curvature R_C and curvature of sign k

curvature

- on a spatial slice (fixed value of t):

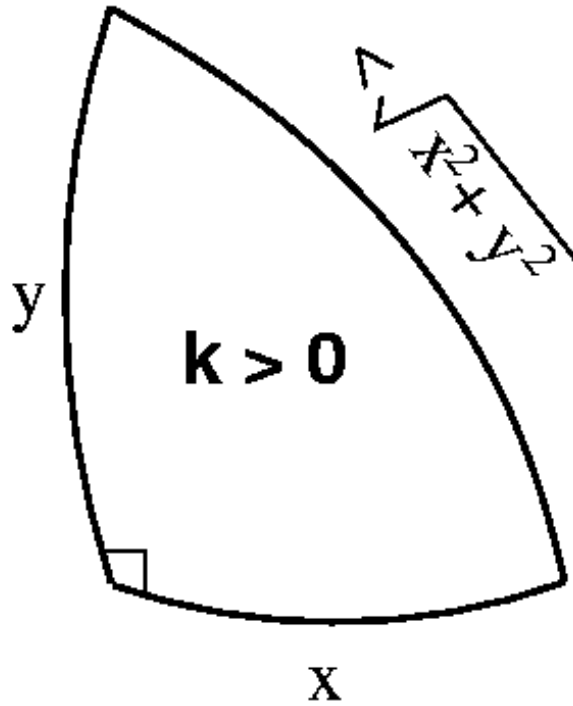


x

$$k = 0$$

curvature

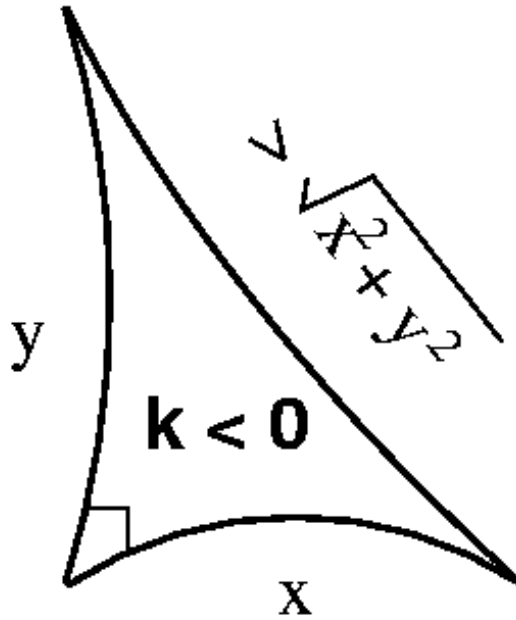
- on a spatial slice (fixed value of t):



$$k > 0$$

curvature

- on a spatial slice (fixed value of t):



$$k < 0$$

2D curvature intuition: $k > 0$

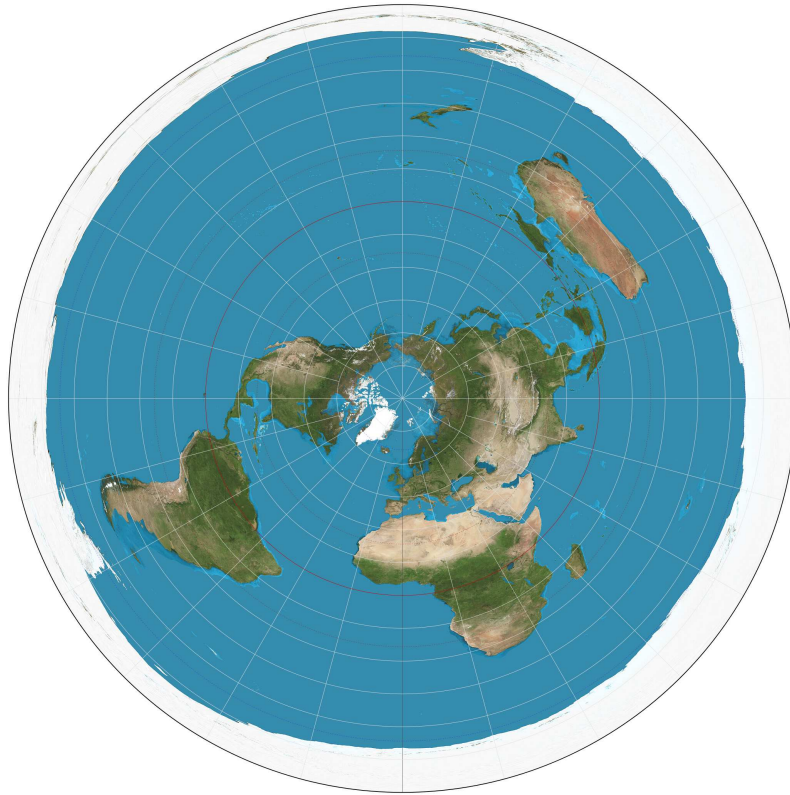
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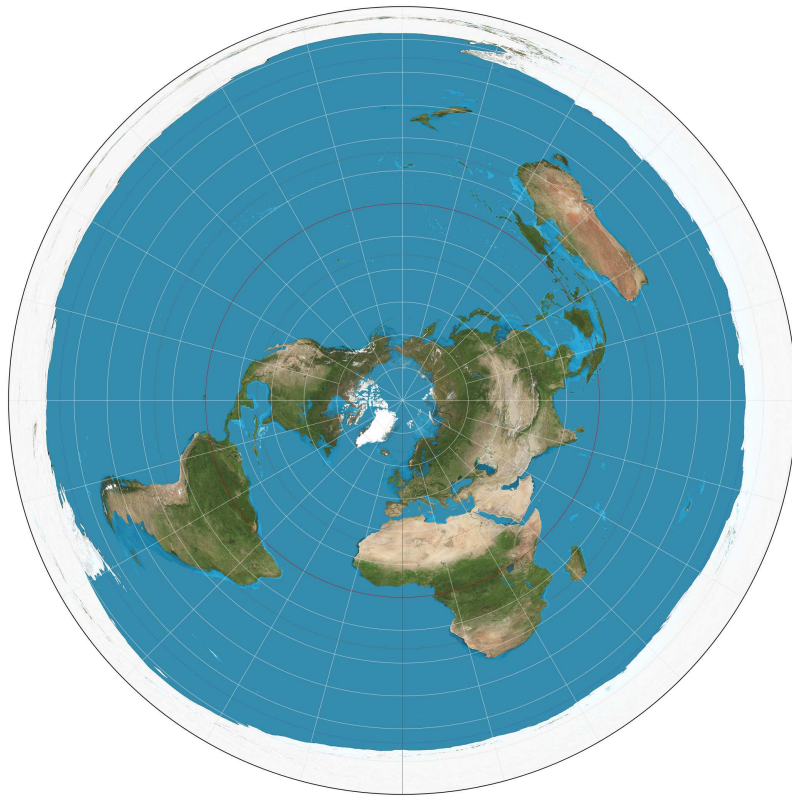


w:

(al-Biruni, c. 1000 CE)

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■ intuition switch: S^2 easier vs S^3 more physical

2D topology intuition ($k = 0$)



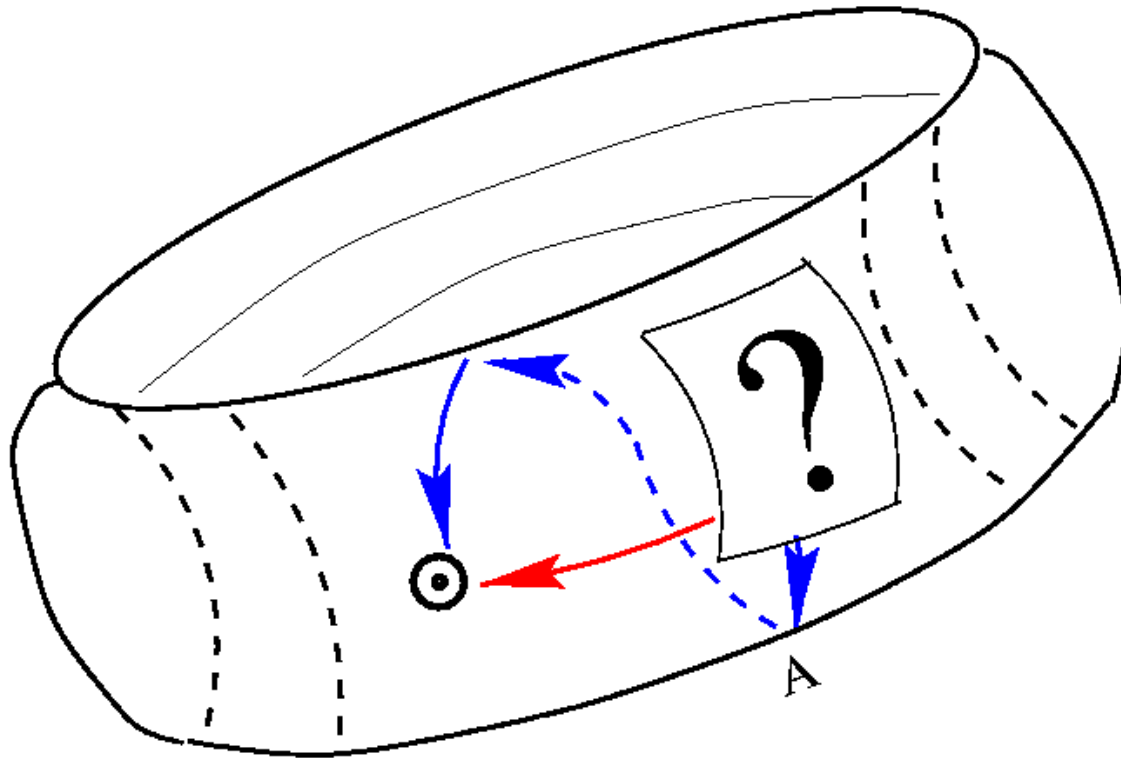
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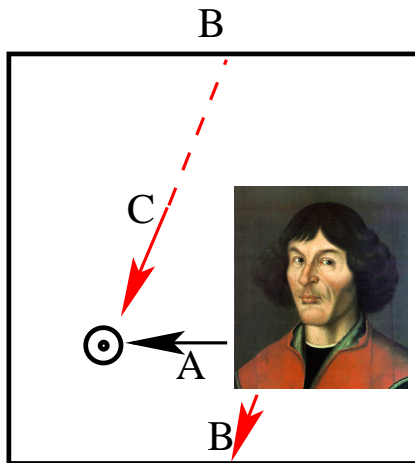


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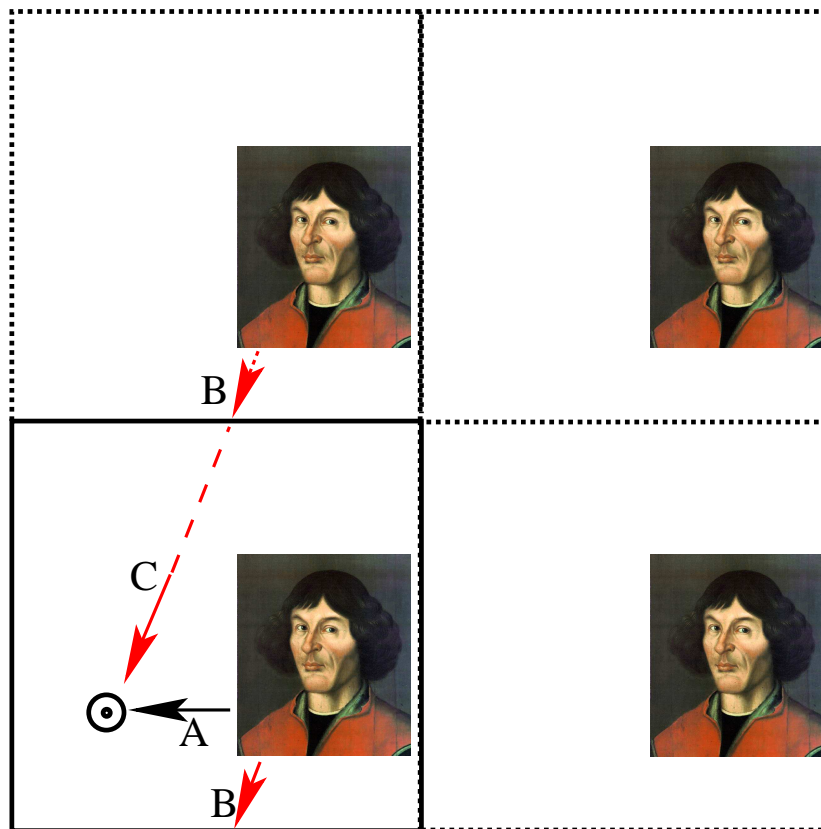


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 $\dot{a}(t_0)/a(t_0) \approx 600 \text{ km/s/Mpc}$

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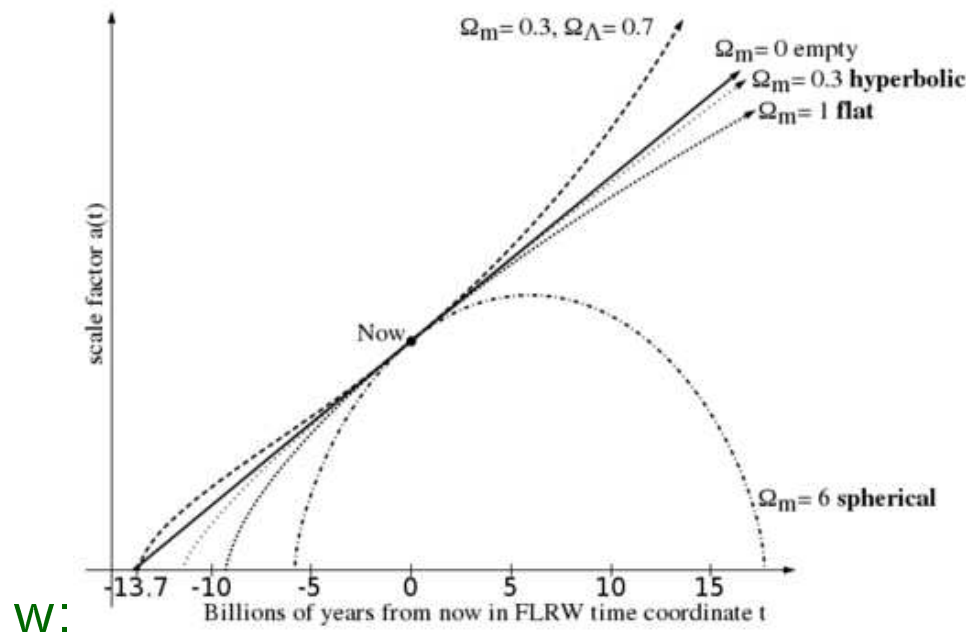
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(Defn: $a_0 := 1$)

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$$1 + z = \frac{1}{a_{\text{em}}}$$

(Defn of redshift z)

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- radiation density: $E = h\nu \Rightarrow \rho_r \propto a^{-4} = (1+z)^4$

Black body: COBE (~ 1992)

■ Planck's Law:
$$I(\nu, T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

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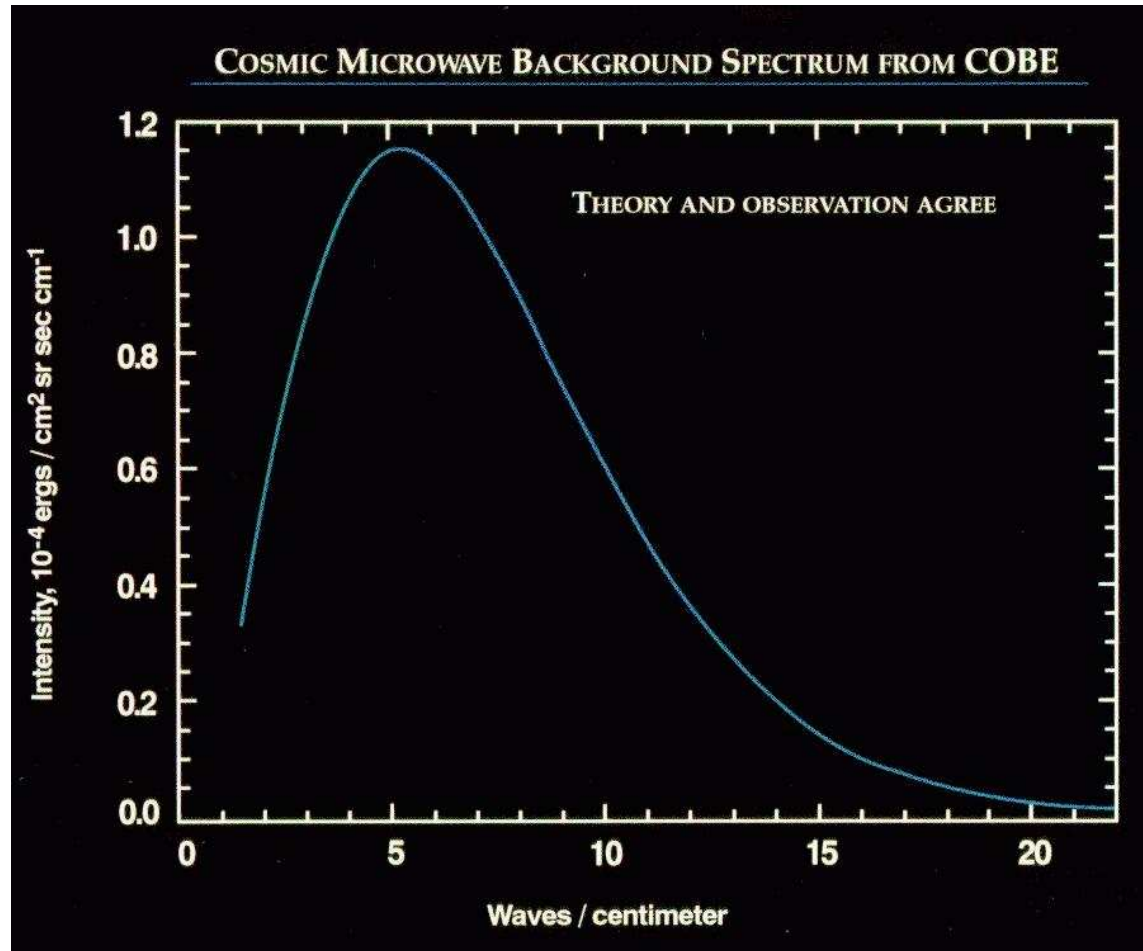
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Black body: COBE (~ 1992)

- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)

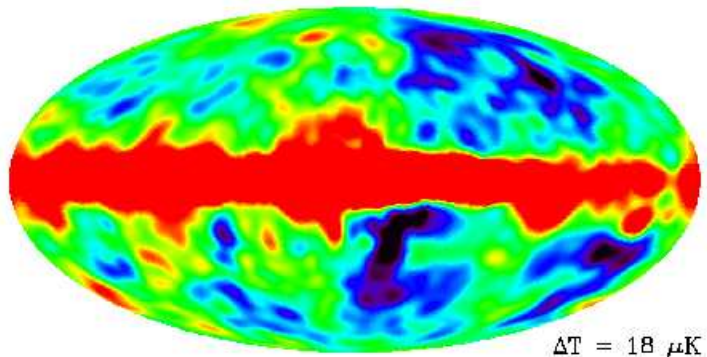
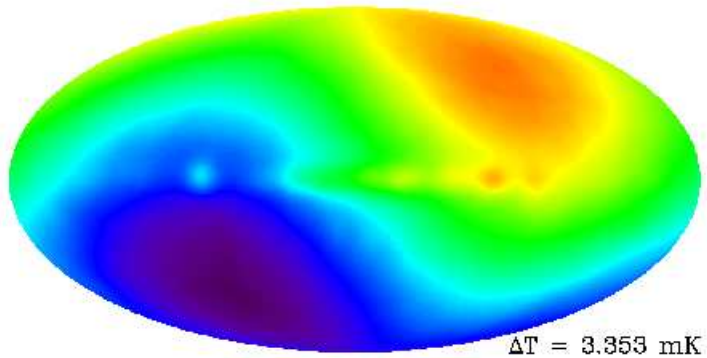
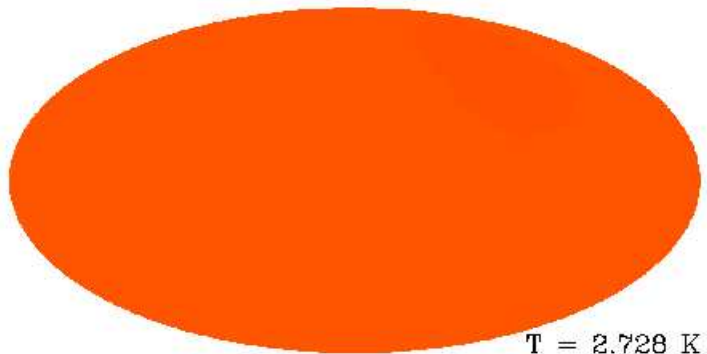
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- COBE /FIRAS (Far Infrared Absolute Spectrophotometer)



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- COBE /DMR (Differential Microwave Radiometer)

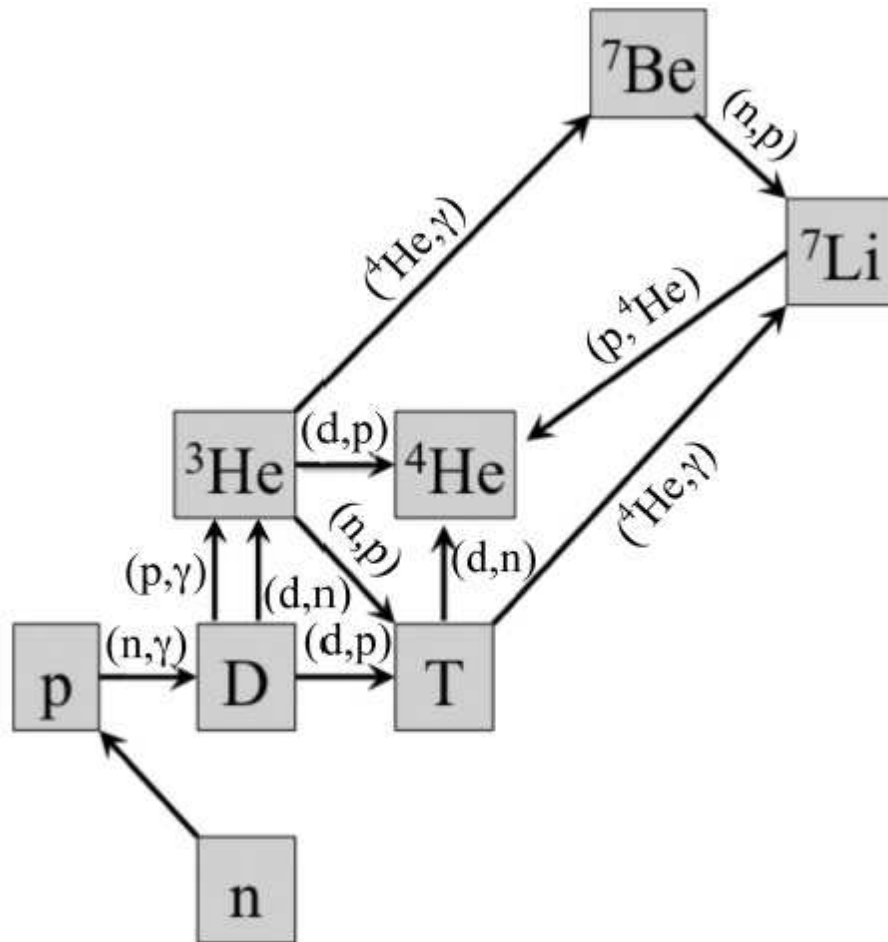


BBN: Big bang nucleosynthesis

- Alpher, Bethe, & Gamow (1948; [ADS:1948PhRv...73..803A](#))

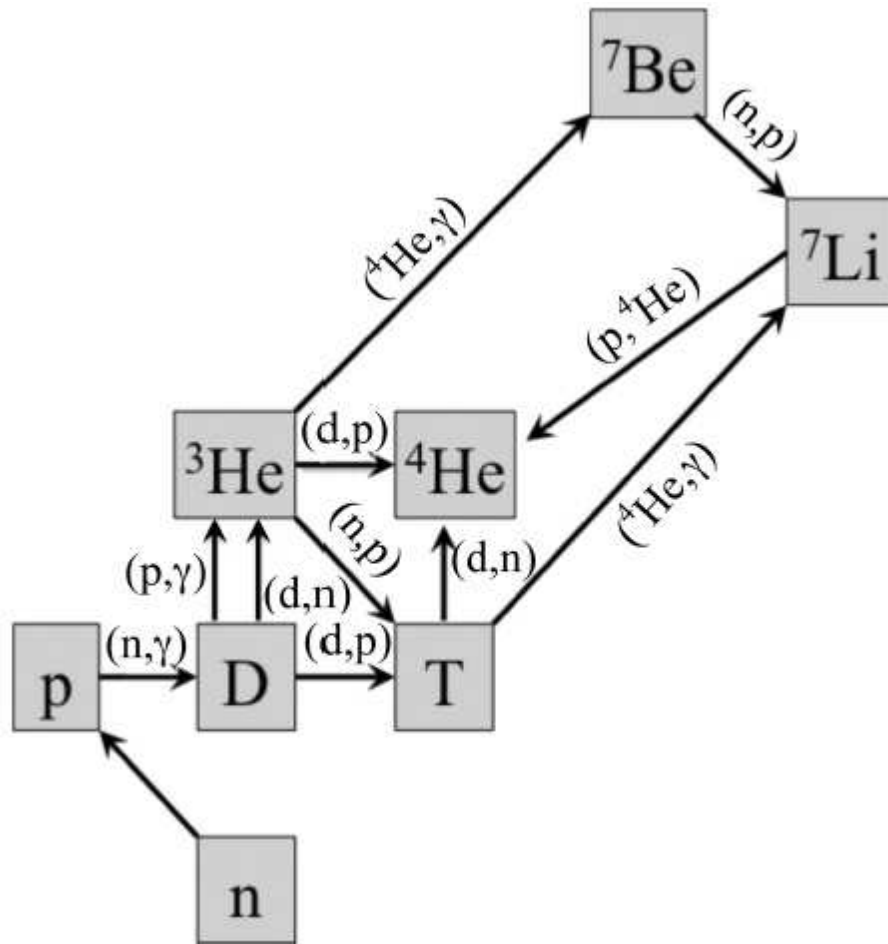
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- acceleration Eqn:
$$\frac{\ddot{a}}{a} = -\frac{4\pi G (\rho + 3p/c^2)}{3}$$

FLRW matter-dominated epoch

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FLRW matter-dominated epoch

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$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

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- $k = 0$ case: $\dot{a}^2 \propto a^{-1}$

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- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$
- $k = 0$ case: $\dot{a} \propto a^{-1/2}$ for $a > 0$

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- $k = 0$ case: $da \propto a^{-1/2} dt$ for $a > 0$

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$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

- matter-dominated epoch: $\rho = \rho_m = \rho_{m0} a^{-3}$

- $k = 0$ case: $a^{1/2} da \propto dt$

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- $k = 0$ case: $(2/3)a^{3/2} \propto t$

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$$\dot{a}^2 = \frac{8\pi G \rho_{m0}}{3a} - c^2 k$$

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- $k = 0$ case:

$$a = \left(\frac{t}{t_0}\right)^{2/3}$$

Einstein–de Sitter model (EdS)

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Einstein–de Sitter model (EdS)

Defn: Hubble parameter $H := \dot{a}/a$

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$\Rightarrow H(t) = \frac{2}{3t}$

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$\Rightarrow H(t) = \frac{2}{3t}; H_0 = H(t_0) = \frac{2}{3t_0}$

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■ convenient conversion: $1 \text{ km/s} \approx 1.04 \text{ kpc/Gyr} \approx 1 \text{ kpc/Gyr}$

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■ Lemaître (1927) [ADS:1927ASSB...47...49L](#): $H_0 \approx 600$ km/s/Mpc

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0}$$

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr}$$

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr} < t_{\text{Earth}} \approx 4.5 \text{ Gyr}$$

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- 1980's: $H_0 \approx 50$ or 100 km/s/Mpc

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- 1980's: $H_0 \approx 0.05$ or 0.1 Gyr^{-1}

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$$\Rightarrow \text{EdS would give } t_0 = \frac{2}{3H_0} \approx 1.3 \text{ Gyr} < t_{\text{Earth}} \approx 4.5 \text{ Gyr}$$

- 1980's: $H_0 \approx 0.05$ or $0.1 \text{ Gyr}^{-1} \Rightarrow t_0(\text{EdS}) \approx 13.0$ or 6.5 Gyr , resp.

FLRW: ρ_{crit}

■ Friedmann Eqn:

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

FLRW: ρ_{crit}

■ Friedmann Eqn:

$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

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$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

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- consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

- ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0$

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- ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0 \text{ flat}$

- ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0$

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- ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0 \text{ spherical}$

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- ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0 \text{ spherical}$

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- ◆ $\rho_{\text{m}0} = \frac{3H_0^2}{8\pi G} \Leftrightarrow k = 0 \text{ flat}$
- ◆ $\rho_{\text{m}0} > \frac{3H_0^2}{8\pi G} \Leftrightarrow k > 0 \text{ spherical}$
- ◆ $\rho_{\text{m}0} < \frac{3H_0^2}{8\pi G} \Leftrightarrow k < 0 \text{ hyperbolic}$

FLRW: ρ_{crit}

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$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

FLRW: ρ_{crit}

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$$H^2 = \frac{8\pi G\rho}{3} - \frac{c^2 k}{a^2}$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

FLRW: ρ_{crit}

■ Friedmann Eqn:
$$H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{c^2 k}{a^2}$$

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$$H^2 = \frac{\rho H^2}{\rho_{\text{crit}}} - \frac{c^2 k}{a^2}$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_{\text{m}} := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

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$$H^2 = \Omega_m H^2 - \frac{c^2 k}{a^2}$$

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$$H^2 = \Omega_m H^2 - \frac{c^2 k}{a^2}$$

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Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

■ Friedmann Eqn: $H^2 = \Omega_m H^2 + \Omega_k H^2$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

FLRW: ρ_{crit}

■ Friedmann Eqn:

$$1 = \Omega_m + \Omega_k$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

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◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$

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◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$

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$$1 = \Omega_m + \Omega_k$$

Defn: $\rho_{\text{crit}} := \frac{3H^2}{8\pi G}$ critical density

Defn: $\Omega_m := \frac{\rho}{\rho_{\text{crit}}}$ matter density parameter

Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ curvature density parameter (sign reversal!)

■ consider a fixed observation, e.g. $H_0 = 100 \text{ km/s/Mpc}$

- ◆ $\Omega_{m0} = 1 \Leftrightarrow k = 0$ flat
- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$

FLRW: ρ_{crit}

■ Friedmann Eqn:

$$1 = \Omega_m + \Omega_k$$

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■ $\Omega_{\text{tot}} := \Omega_m +$

FLRW: ρ_{crit}

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■ $\Omega_{\text{tot}} := \Omega_{\text{m}} + \Omega_{\text{r}} +$

FLRW: ρ_{crit}

■ Friedmann Eqn:

$$1 = \Omega_{\text{tot}} + \Omega_k$$

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■ $\Omega_{\text{tot}} := \Omega_m + \Omega_r + \Omega_\Lambda$

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- ◆ $\Omega_{m0} > 1 \Leftrightarrow k > 0$ spherical
- ◆ $\Omega_{m0} < 1 \Leftrightarrow k < 0$ hyperbolic

■ $\Omega_{\text{tot}} := \Omega_b + \Omega_{\text{nbDM}} + \Omega_r + \Omega_\Lambda$

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

FLRW curvature constant

■ metric in

- ◆ azimuthal equidistant coords: R_C
- ◆ orthographic coords: k

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

FLRW curvature constant

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◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_C^2 = 1$

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$

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FLRW curvature constant

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◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$

■ Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow \Omega_{k0} = -\frac{c^2 k}{H_0^2}$

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$

■ Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow k = -\frac{\Omega_{k0} H_0^2}{c^2}$

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

■ $\Rightarrow kR_C^2 = 1 \Rightarrow k = 1/R_C^2$

■ Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2}$ $\Rightarrow R_C^2 = -\frac{c^2}{\Omega_{k0} H_0^2}$

FLRW curvature constant

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■ Defn: $\Omega_k := -\frac{c^2 k}{a^2 H^2} \Rightarrow R_C^2 = -\frac{c^2}{H_0^2} \frac{1}{1 - \Omega_{\text{tot}0}}$

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■ $\Omega_{\text{tot}0} > 1$

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■ $\Omega_{\text{tot}0} > 1$ *spherical*

FLRW curvature constant

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■ $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

FLRW curvature constant

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■ $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

■ $\Omega_{\text{tot}0} = 1$

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◆ azimuthal equidistant coords: R_C

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■ $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

■ $\Omega_{\text{tot}0} = 1$ *flat*

FLRW curvature constant

■ metric in

◆ azimuthal equidistant coords: R_C

◆ orthographic coords: k

■ orthographic: $1 - kr^2 = 0$ coord singularity at equator

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■ $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

■ $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined

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FLRW curvature constant

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■ $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined

■ $\Omega_{\text{tot}0} < 1$ *hyperbolic*

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■ $\Omega_{\text{tot}0} > 1$ *spherical* R_C real

■ $\Omega_{\text{tot}0} = 1$ *flat* R_C undefined

■ $\Omega_{\text{tot}0} < 1$ *hyperbolic* R_C imaginary (or use $|R_C|$)

Einstein's free parameter: Λ

- Einstein: prevent expansion/contraction via Λ
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[ADS:1917SPAW.....142E](#)
- MAXIMA: calculate \mathbf{G} and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
<http://cosmo.torun.pl/Cosmo/FLRWEquationsGR>

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- MAXIMA: calculate \mathbf{G} and $\mathbf{G} - \mathbf{g}\Lambda = 8\pi\mathbf{T}$ and simplify:
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- *hint*: mixed index form of \mathbf{g} is easy

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Friedmann Eqn ($\Lambda \neq 0$):

$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{H^2}{2} \frac{\rho}{\rho_{\text{crit}}} + \Omega_\Lambda H^2$$

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Friedmann Eqn ($\Lambda \neq 0$):

$$\frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a}}{a} = -\frac{H^2 \Omega_m}{2} + \Omega_\Lambda H^2$$

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ADS:1917SPAW.....142E

$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

Defn: "dust solution": $p(t) = 0$

$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad \frac{\ddot{a} a^2}{a \dot{a}^2} = -\frac{\Omega_m}{2} + \Omega_\Lambda$$

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Defn: "dust solution": $p(t) = 0$

Defn: $\Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$

acceleration Eqn ($\Lambda \neq 0$):

$$\frac{\ddot{a} a^2}{a \dot{a}^2} = -\frac{\Omega_m}{2} + \Omega_\Lambda$$

Defn: $q := -\frac{\ddot{a} a}{\dot{a}^2}$

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Defn: “dust solution”: $p(t) = 0$

$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

$$\text{acceleration Eqn } (\Lambda \neq 0): \quad q = \frac{\Omega_m}{2} - \Omega_\Lambda$$

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$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

- $\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Lambda = 0$ and $\Omega_m > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

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- $\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Lambda = 0$ and $\Omega_m > 0$ then $\frac{\ddot{a}}{a} < 0$, i.e. $q > 0$ *deceleration*

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$$\text{Friedmann Eqn } (\Lambda \neq 0): \quad \frac{c^2 k}{a^2} + \frac{\dot{a}^2}{a^2} = \frac{8 \pi G \rho}{3} + \frac{c^2 \Lambda}{3}$$

Defn: “dust solution”: $p(t) = 0$

$$\text{Defn: } \Omega_\Lambda := \frac{c^2 \Lambda}{3 H^2}$$

Defn: $q := -\frac{\ddot{a}a}{\dot{a}^2}$ “deceleration parameter”

- $\boxed{q = \frac{\Omega_m}{2} - \Omega_\Lambda}$ acceleration equation

- if $\Omega_\Lambda > \Omega_m/2$ then $\frac{\ddot{a}}{a} > 0$, i.e. $q < 0$ *acceleration*

distances in FLRW cosmology

- azimuthal equidistant r :

distances in FLRW cosmology

- azimuthal equidistant r : proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c dt'}{a(t')}$

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distances in FLRW cosmology

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- Friedmann Eq: $1 = \Omega_m + \Omega_k + \Omega_\Lambda$
- Ω_m = $\frac{\rho}{\rho_{\text{crit}}}$

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- $\Omega_m = \frac{\rho}{\rho_{\text{crit}}} = \frac{\rho_0 a^{-3}}{\rho_{\text{crit}0} (H^2/H_0^2)}$

distances in FLRW cosmology

- azimuthal equidistant r : proper distance at $t_0 \equiv$ comoving radial distance $r = \int_t^{t_0} \frac{c dt'}{a(t')}$
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- $\Omega_m = \frac{\rho}{\rho_{\text{crit}}} = \frac{\rho_0 a^{-3}}{\rho_{\text{crit}0} (H^2/H_0^2)} = \Omega_{m0} H_0^2 a^{-1} \dot{a}^{-2}$

distances in FLRW cosmology

- azimuthal equidistant r : proper distance at $t_0 \equiv$ comoving radial

$$\text{distance } r = \int_t^{t_0} \frac{c dt'}{a(t')}$$

- Friedmann Eq: $1 = \Omega_m + \Omega_k + \Omega_\Lambda$

- $\Omega_m = \frac{\rho}{\rho_{\text{crit}}} = \frac{\rho_0 a^{-3}}{\rho_{\text{crit}0} (H^2/H_0^2)} = \Omega_{m0} H_0^2 a^{-1} \dot{a}^{-2}$

- $\Omega_k = \Omega_{k0} \dot{a}^{-2} H_0^2$

distances in FLRW cosmology

- azimuthal equidistant r : proper distance at $t_0 \equiv$ comoving radial

$$\text{distance } r = \int_t^{t_0} \frac{c dt'}{a(t')}$$

- Friedmann Eq: $1 = \Omega_m + \Omega_k + \Omega_\Lambda$

- $\underline{\Omega_m} = \frac{\rho}{\rho_{\text{crit}}} = \frac{\rho_0 a^{-3}}{\rho_{\text{crit}0} (H^2/H_0^2)} = \Omega_{m0} H_0^2 a^{-1} \dot{a}^{-2}$

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EdS: radial comoving distance

$$\blacksquare r = \int_{1/(1+z)}^1 \frac{c}{H_0} \frac{da}{a \sqrt{\Omega_{m0} a^{-1} + \Omega_{k0} + \Omega_{\Lambda 0} a^2}}$$

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- high-level frontends (e.g. python) should be easy to write

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- r_{\perp} =: “proper motion” distance (Weinberg 1972)

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- $d_A = r_{\perp} a$ (scaled by the scale factor)

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luminosity distance

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- $E = h\nu$ of photon: emitter frame to observer frame
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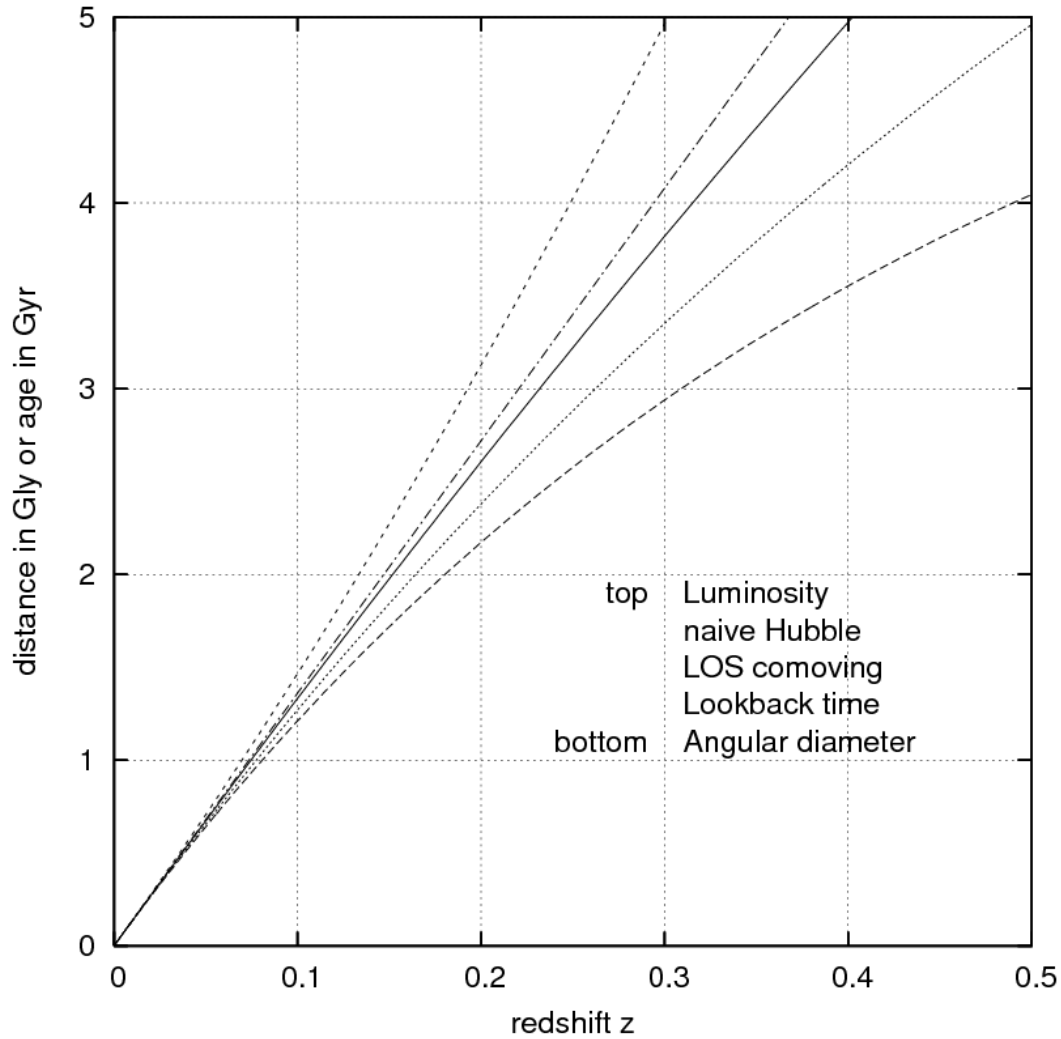
- w:Distance measures (cosmology)



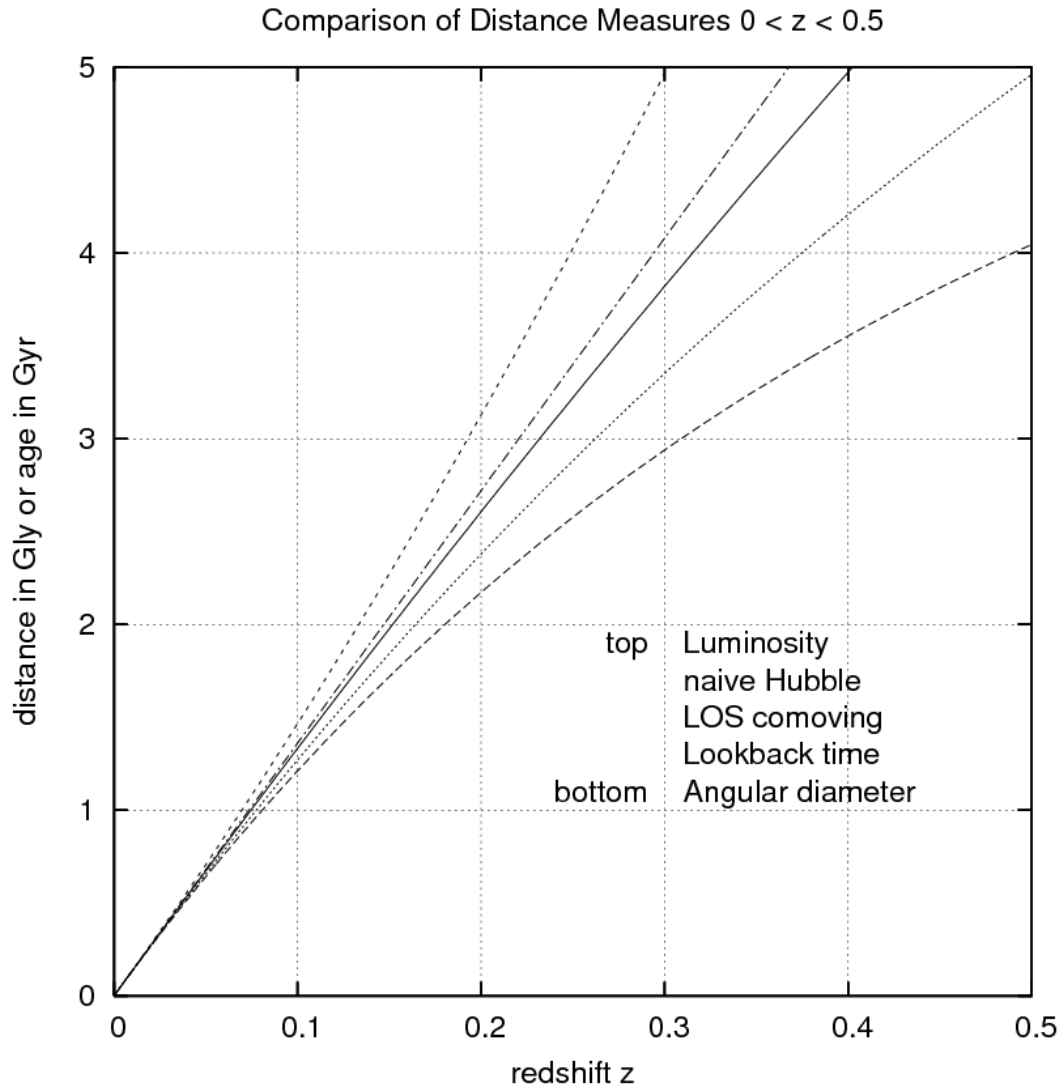
FLRW distances e.g. Λ CDM



Comparison of Distance Measures $0 < z < 0.5$

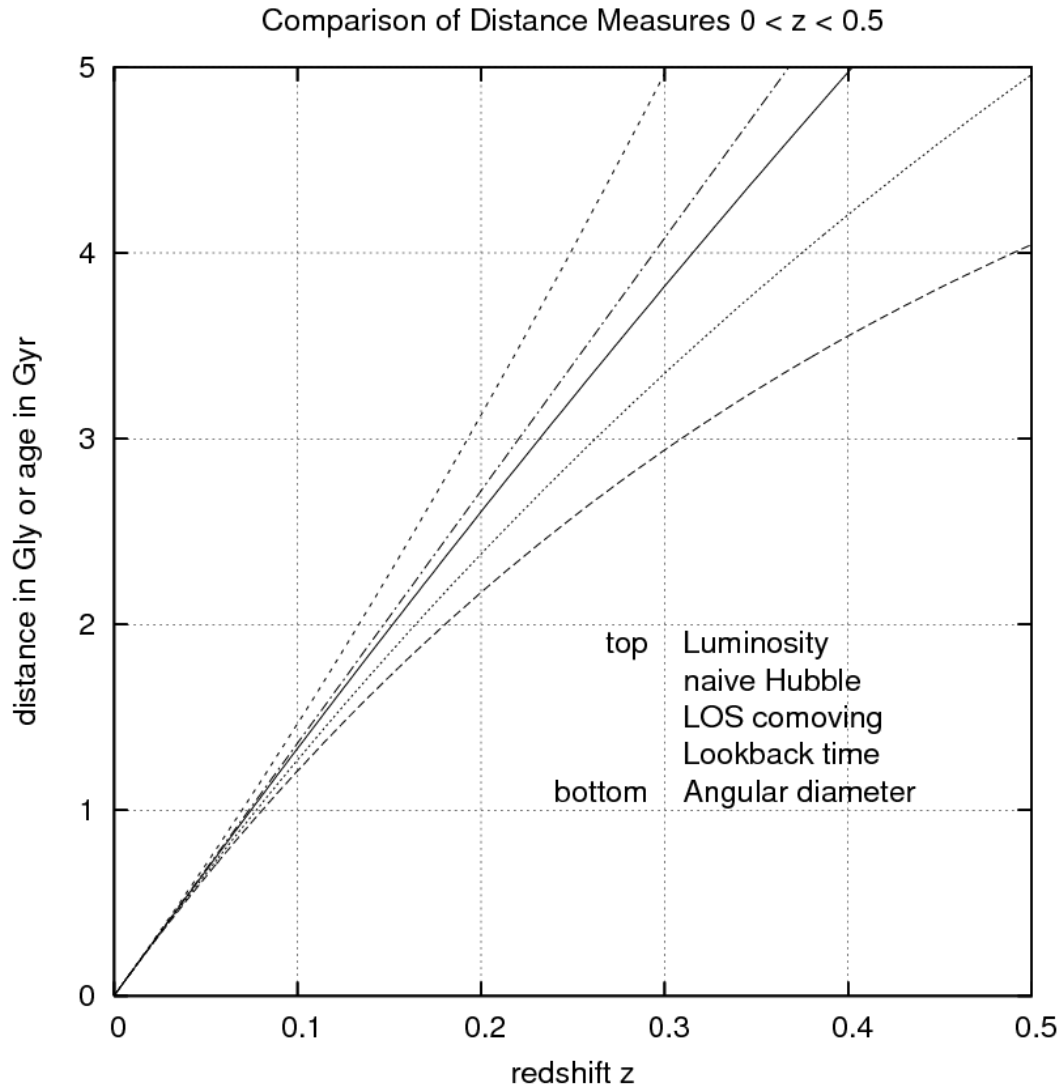


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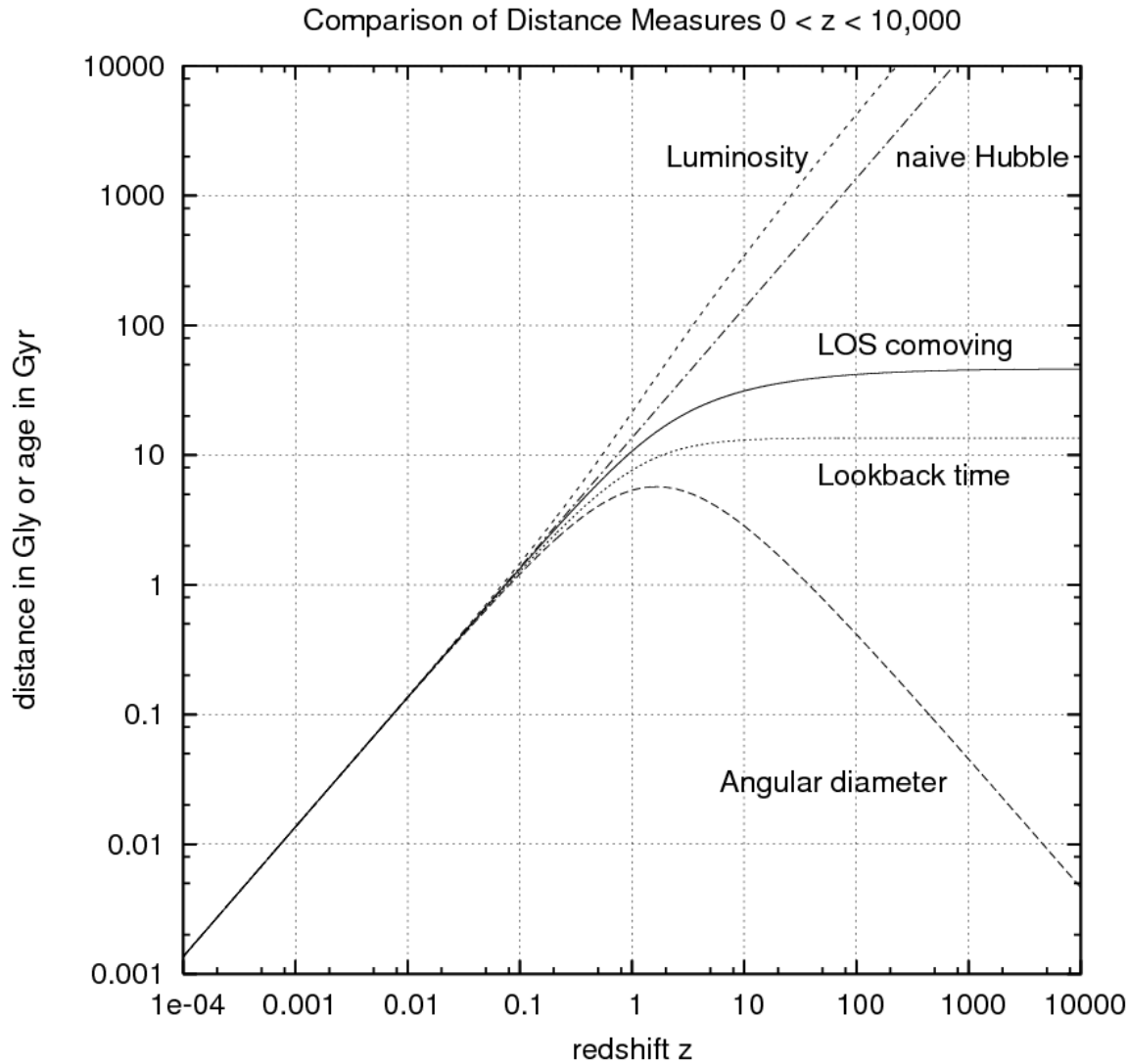
Defn: $h := H_0/100 \text{ km/s/Mpc}$ (without a "0" subscript on h)

FLRW distances e.g. Λ CDM



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\Rightarrow no conflict with locally Lorentzian (SR) spacetime

Non-radial spatial geodesics

- What is the comoving distance between two objects at different celestial positions and different redshifts, for an arbitrary curvature (+, 0, -)?

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- distances on the 2-sphere, embedded in \mathbb{R}^3

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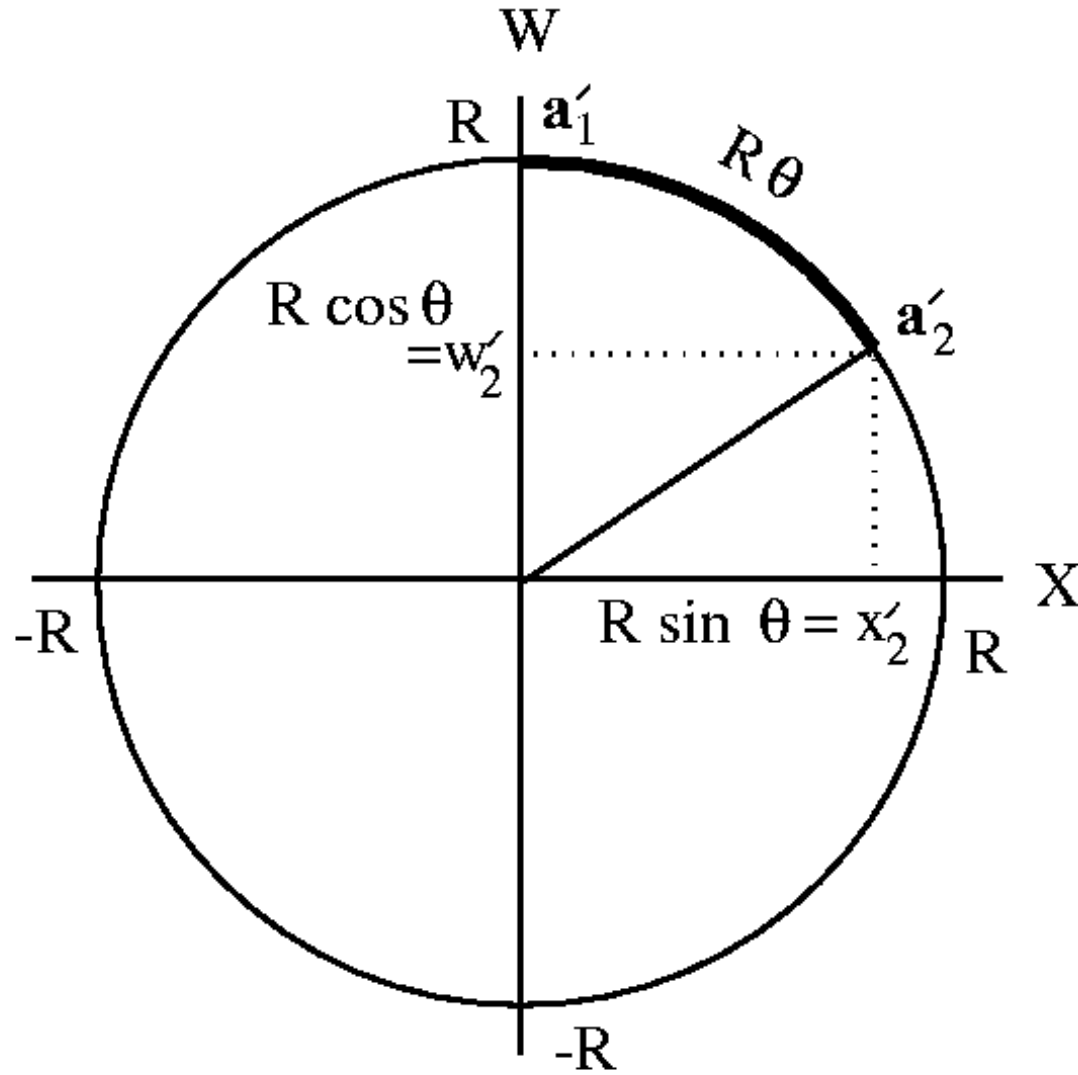
$$\chi_{12} = R_C \theta_{12} = R_C \cos^{-1} [\langle \mathbf{a}_1, \mathbf{a}_2 \rangle / R_C^2]$$

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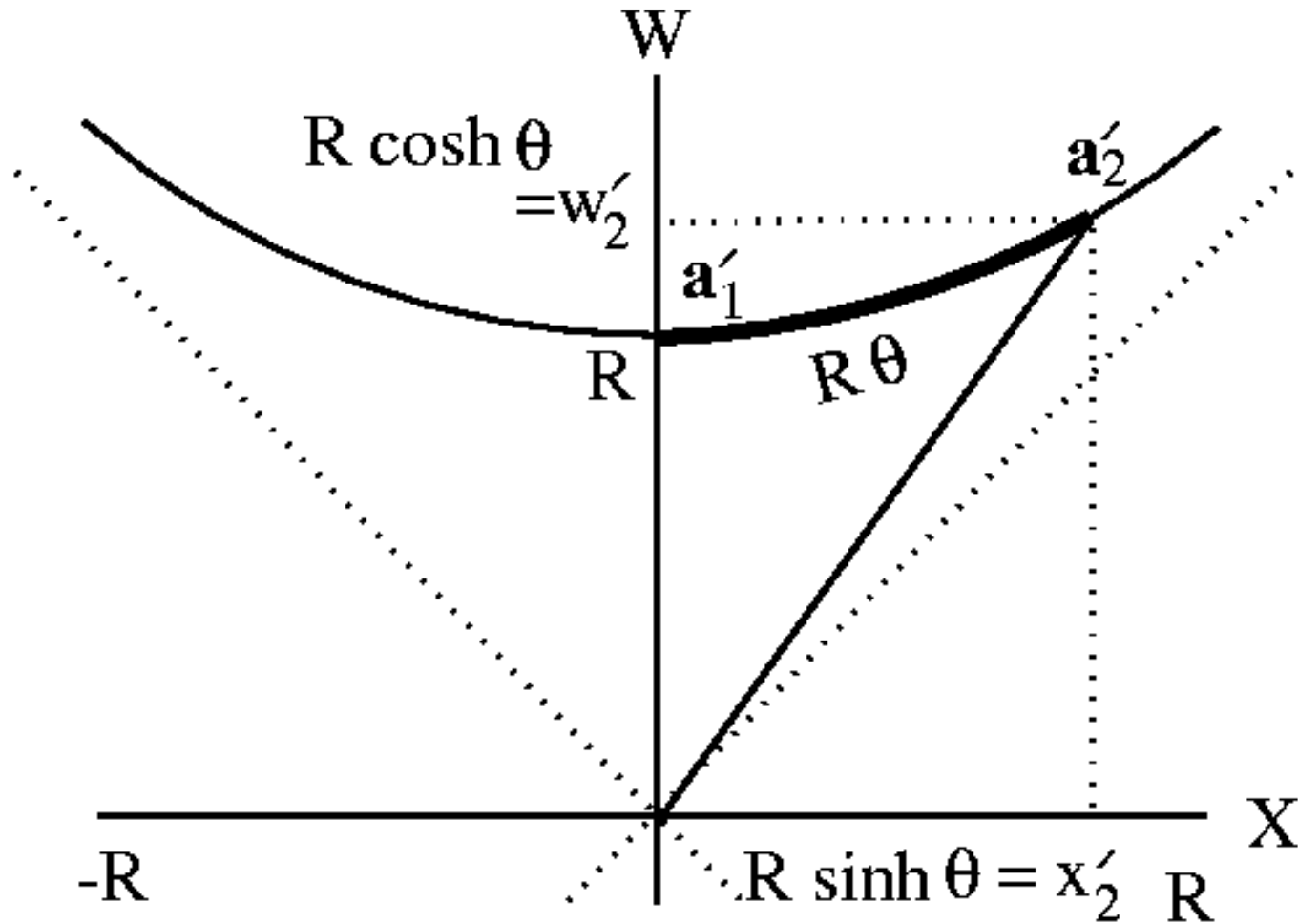


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■ metric on S^3 (or \mathbb{R}^3 or H^3):

$$ds^2 = \begin{cases} (k/|k|) (dx^2 + dy^2 + dz^2) + dw^2 & k \neq 0 \\ dx^2 + dy^2 + dz^2 & k = 0 \end{cases}$$

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inner product:

$$\langle \mathbf{a}_1, \mathbf{a}_2 \rangle \equiv \begin{cases} (k/|k|) (x_1x_2 + y_1y_2 + z_1z_2) + w_1w_2 & k \neq 0 \\ x_1x_2 + y_1y_2 + z_1z_2 & k = 0 \end{cases}$$

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$$w_i = \begin{cases} R_C \cosh(\chi_i/R_C) & k < 0 \\ 0 & k = 0 \\ R_C \cos(\chi_i/R_C) & k > 0 \end{cases}$$

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distances on $S^3 \subset \mathbb{R}^4$ or $H^3 \subset M^4$

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■ a distance in S^3 is an arc-length in \mathbb{R}^4 [arXiv:astro-ph/0102099](https://arxiv.org/abs/astro-ph/0102099)

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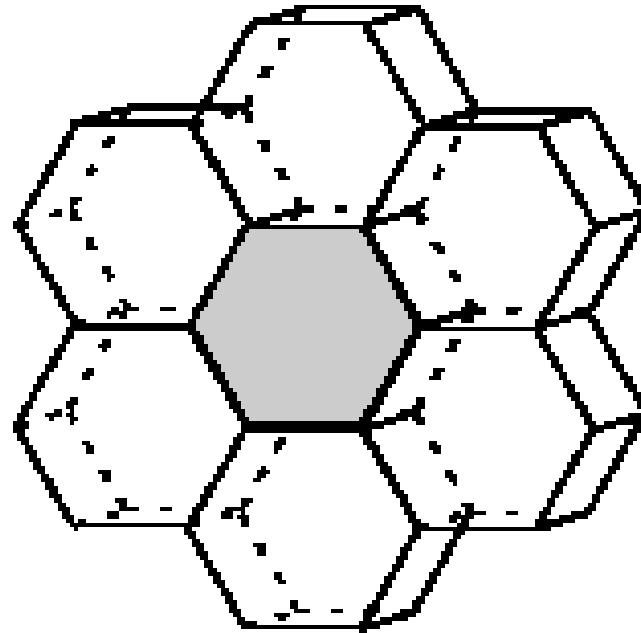
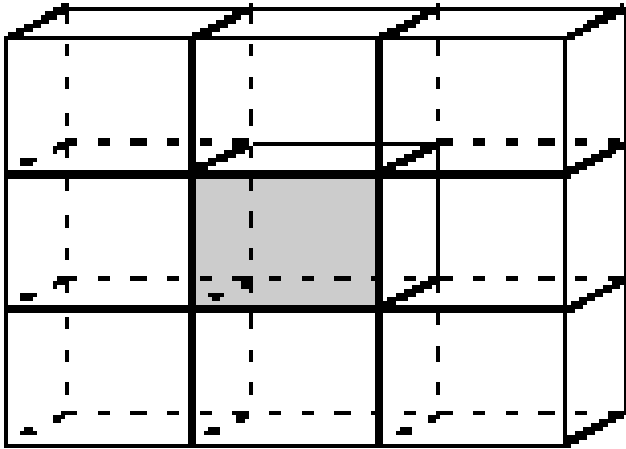
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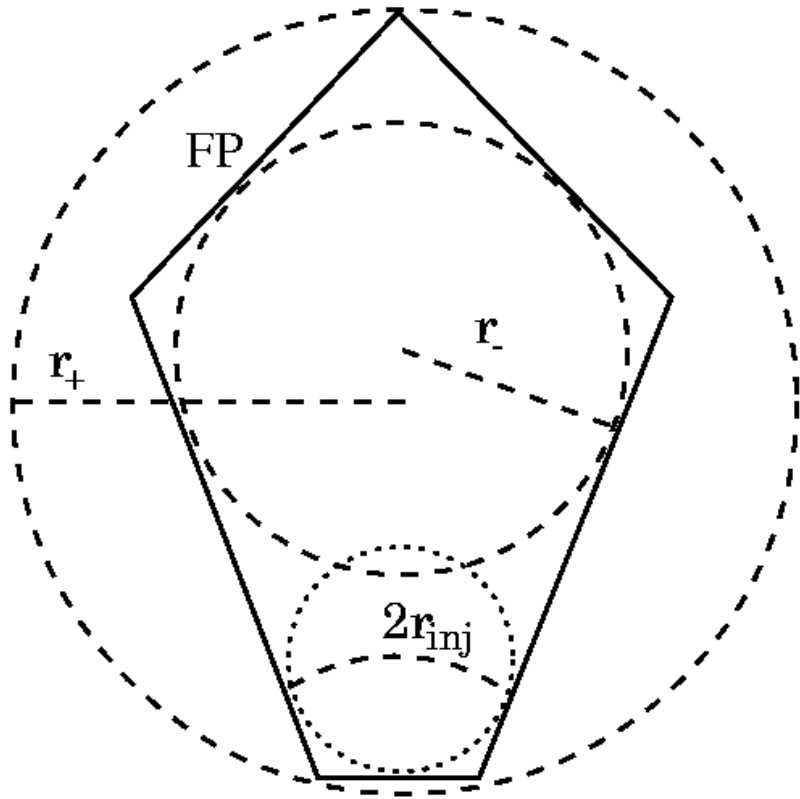
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Cosmic topology: definitions



3D flat examples [arXiv:astro-ph/9901364](https://arxiv.org/abs/astro-ph/9901364)

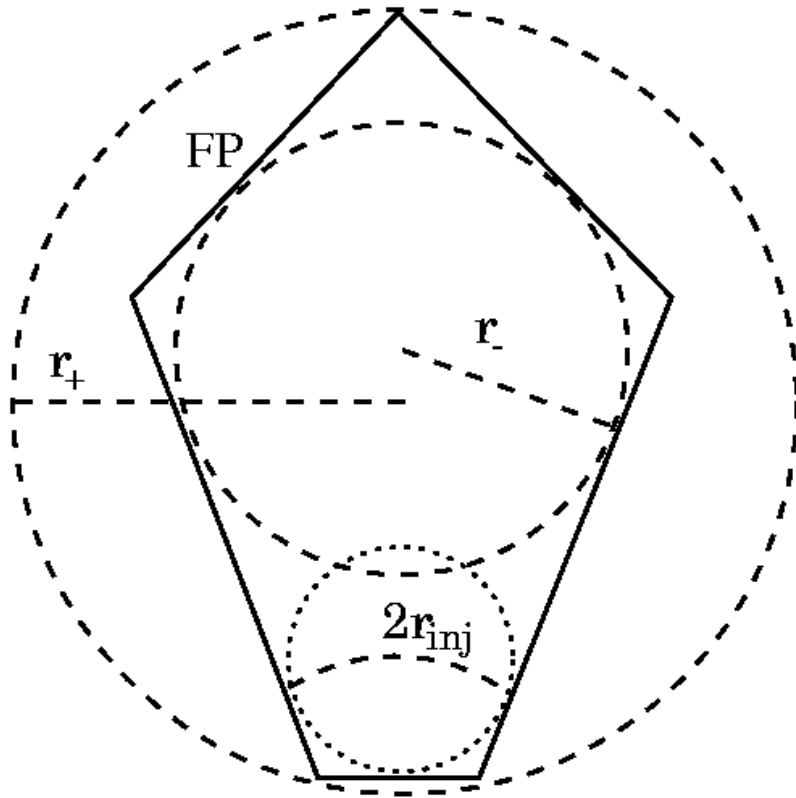
Cosmic topology: definitions



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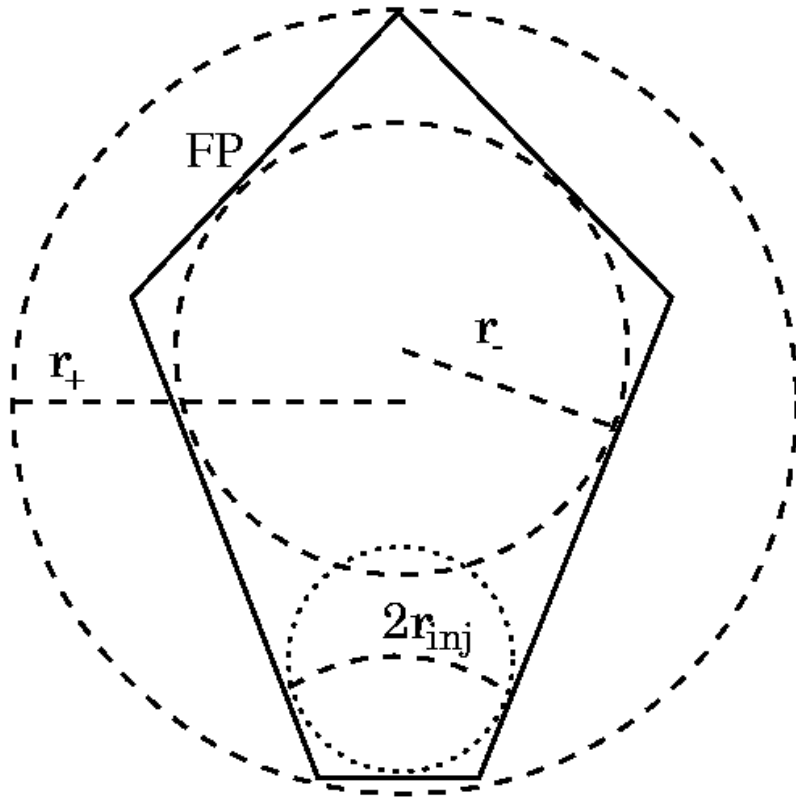
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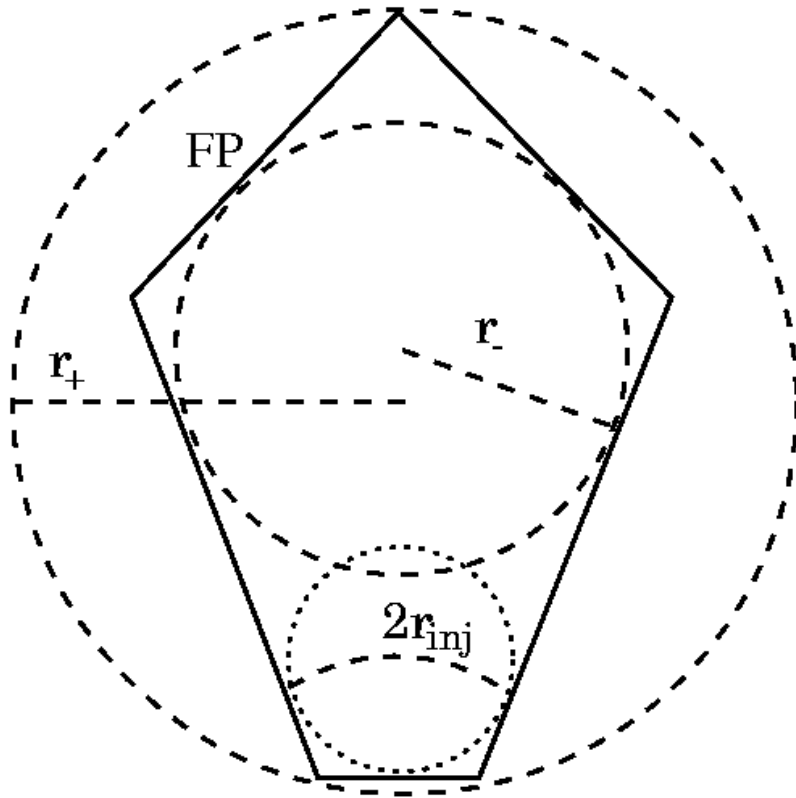
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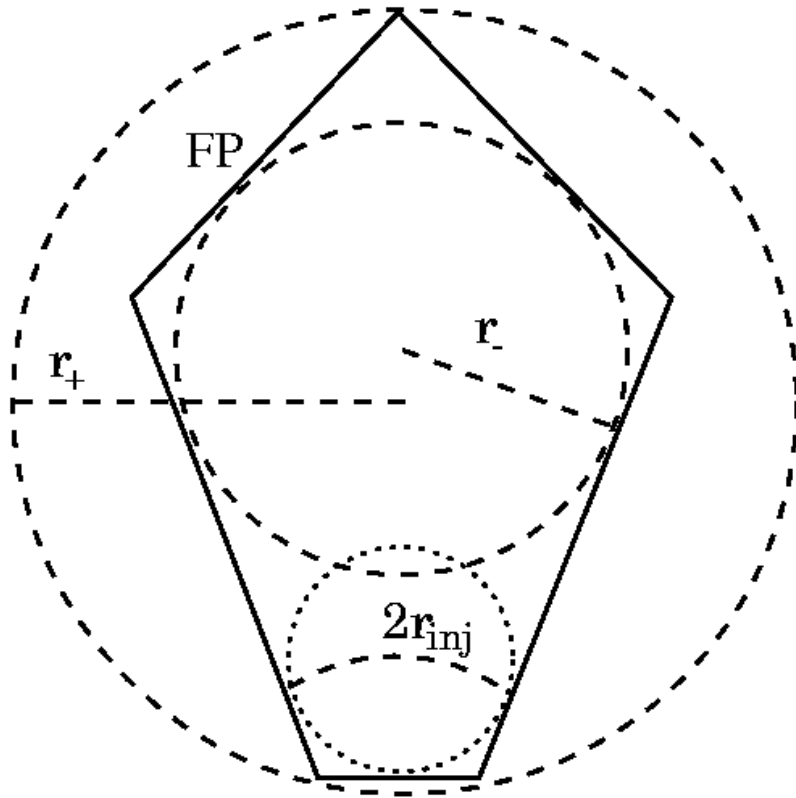
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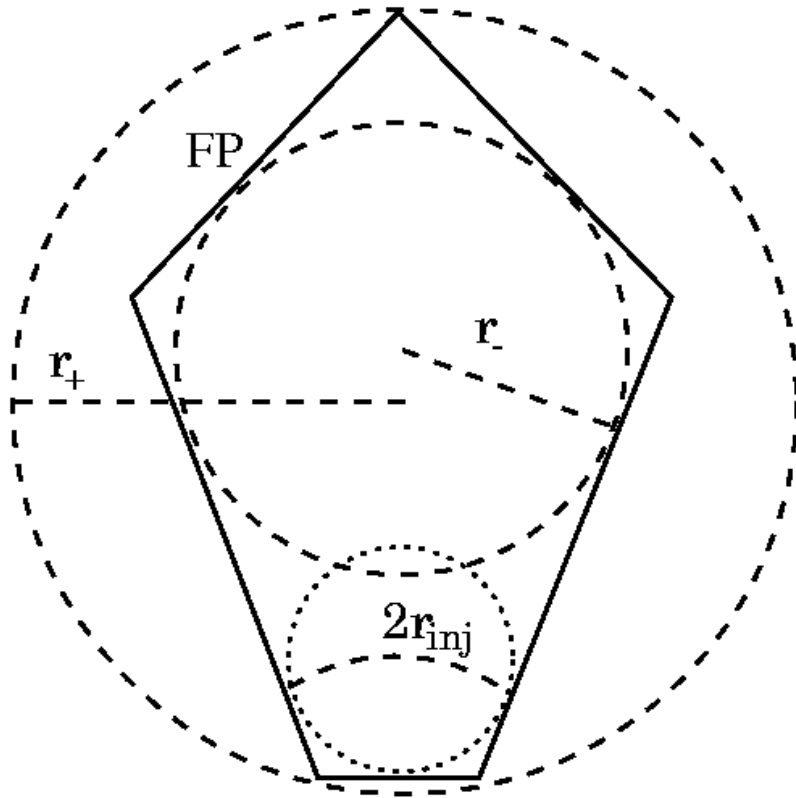
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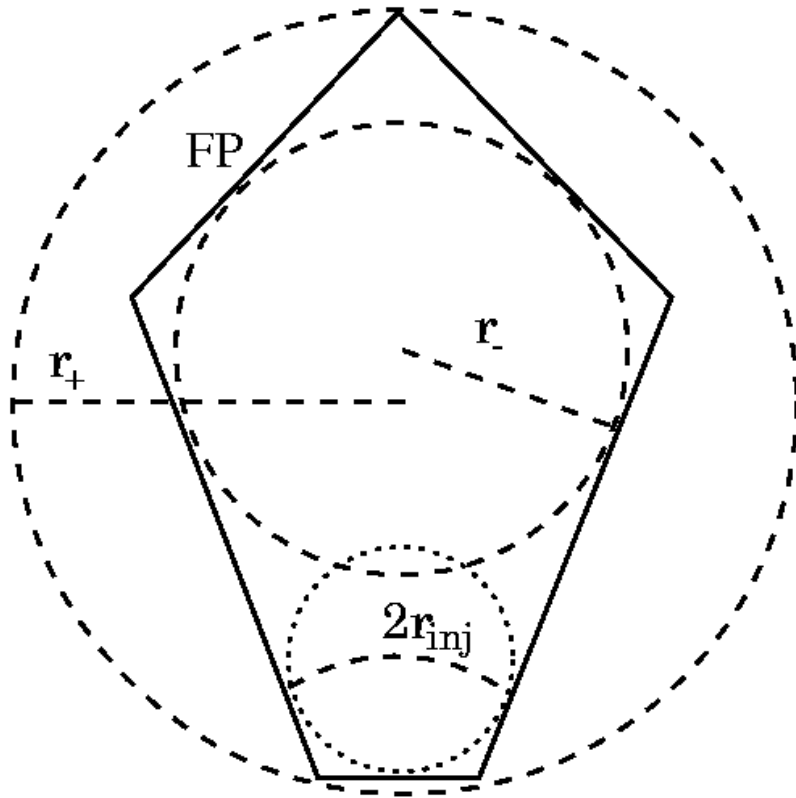
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- $r_{\text{inj}} < r_-$ or $r_{\text{inj}} \ll V_{\text{FD}}^{1/3}$ possible





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 - ◆ w:Poincaré Conjecture “Every simply connected, closed 3-manifold is homeomorphic to the 3-sphere.”
w:Grigori Perelman, [arXiv:math/0211159](https://arxiv.org/abs/math/0211159) + [arXiv:math/0303109](https://arxiv.org/abs/math/0303109) + [arXiv:math/0307245](https://arxiv.org/abs/math/0307245)

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Cosmic topol: top. accel.

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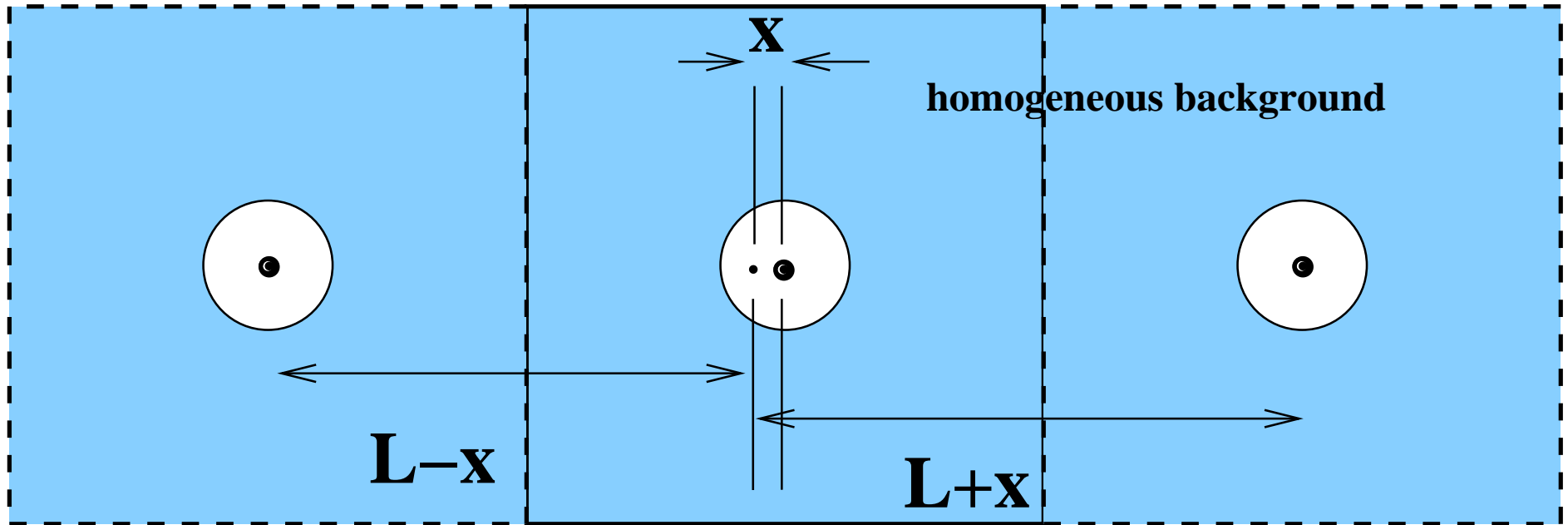
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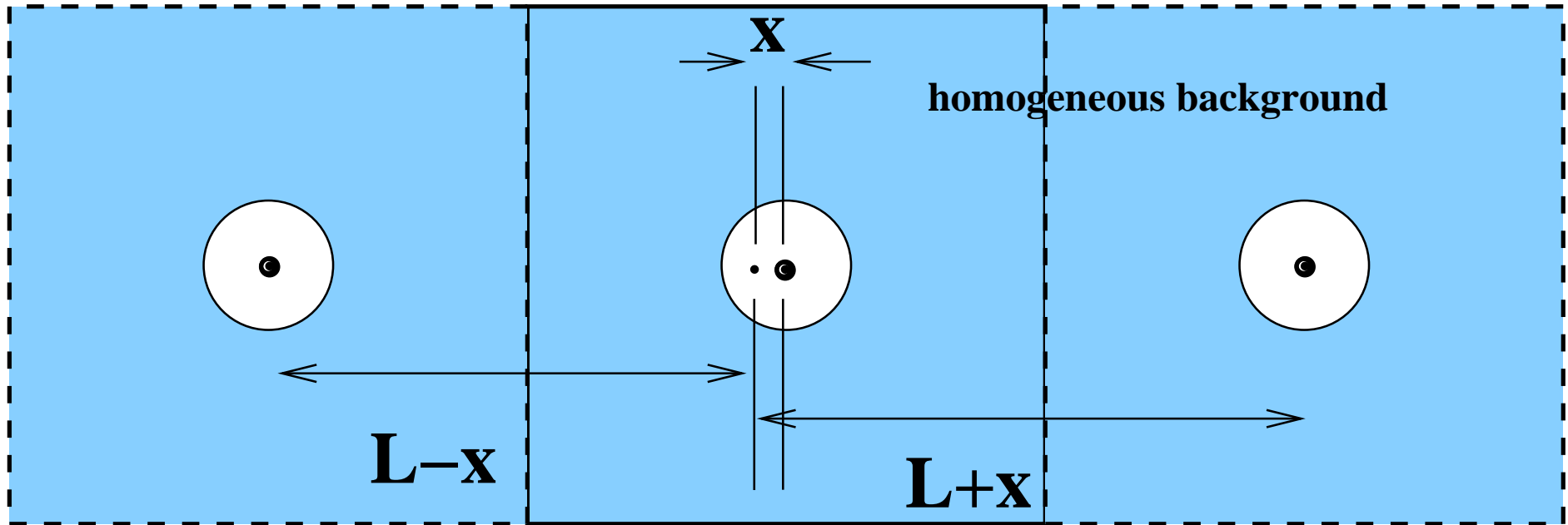
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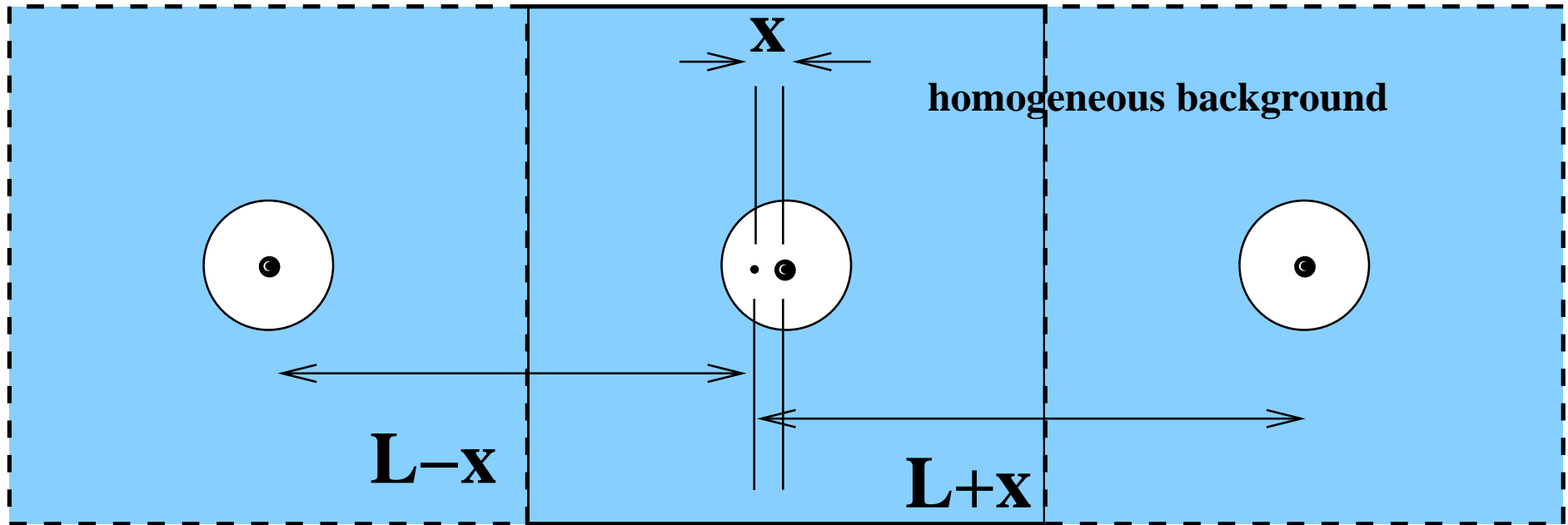


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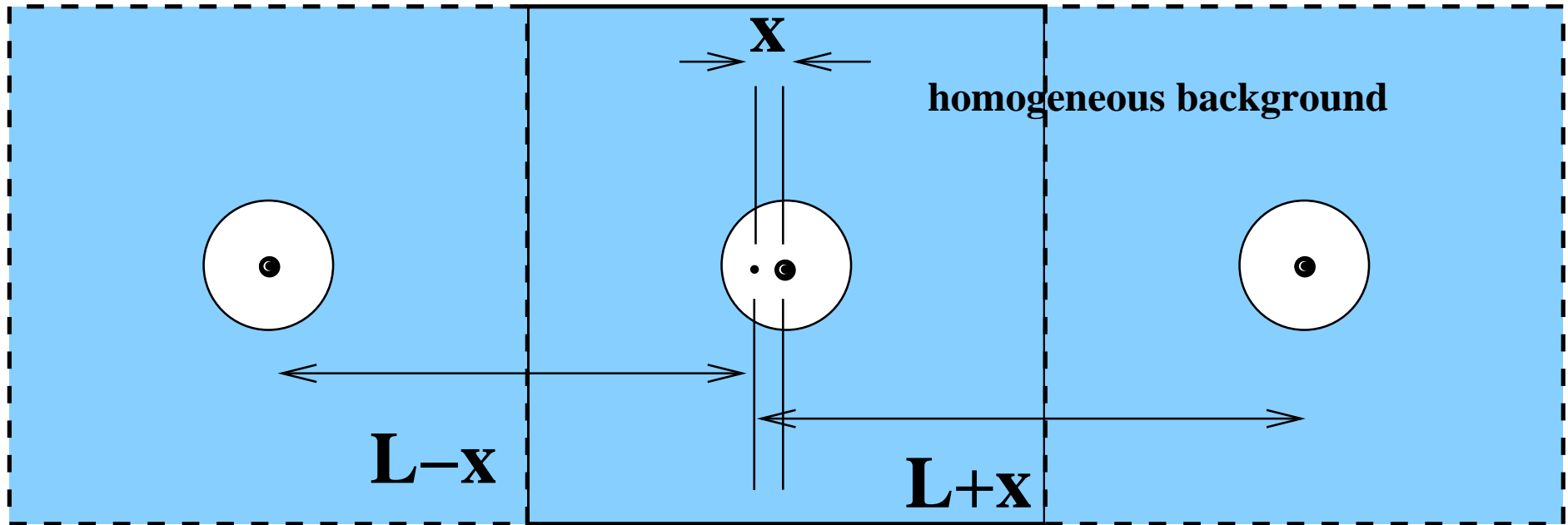
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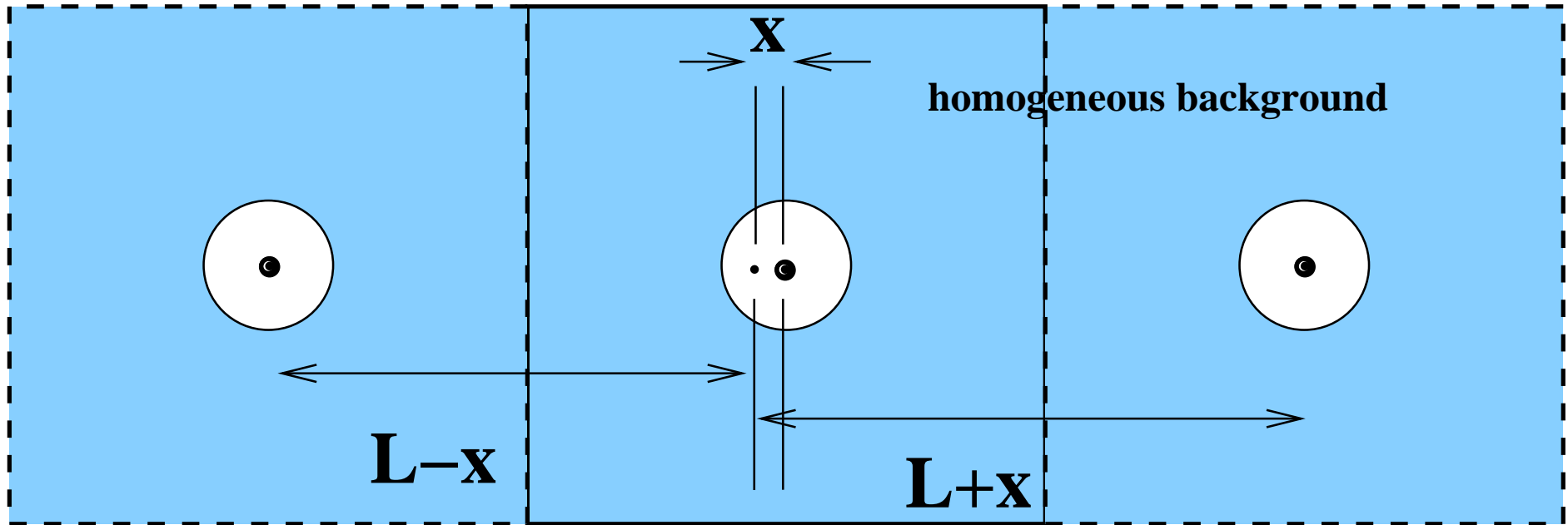


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topological acceleration— [arXiv:astro-ph/0602159](https://arxiv.org/abs/0602159)

Cosmic topol: obs. strategies

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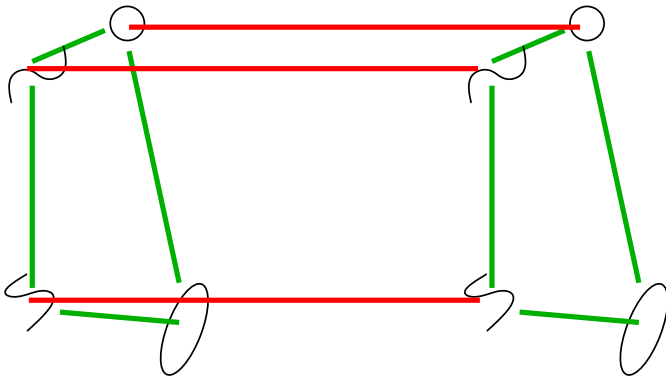
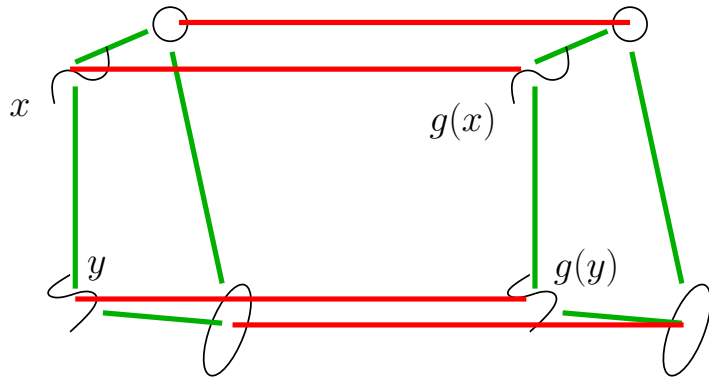
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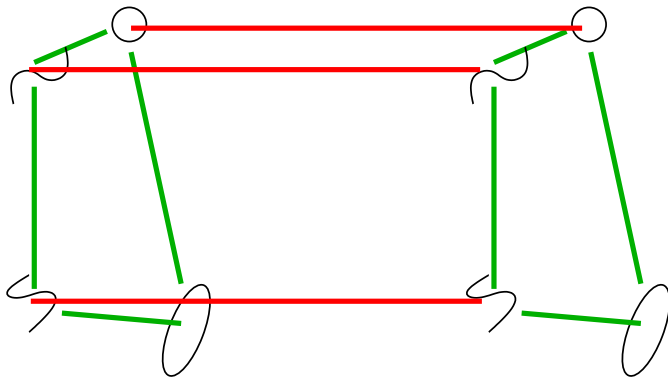
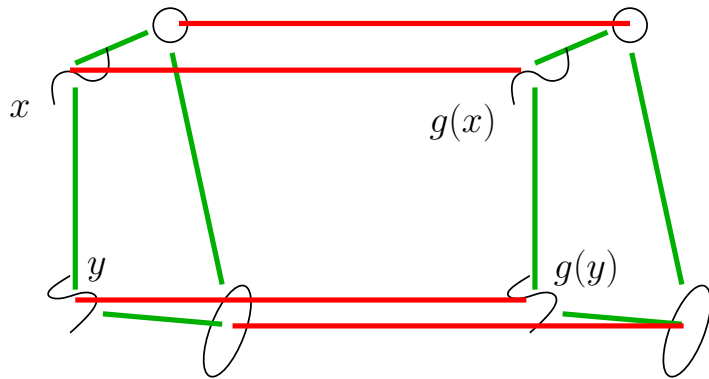
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3D strategies—pair types

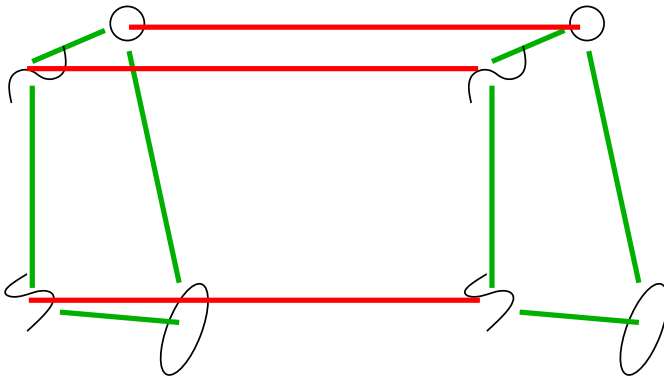
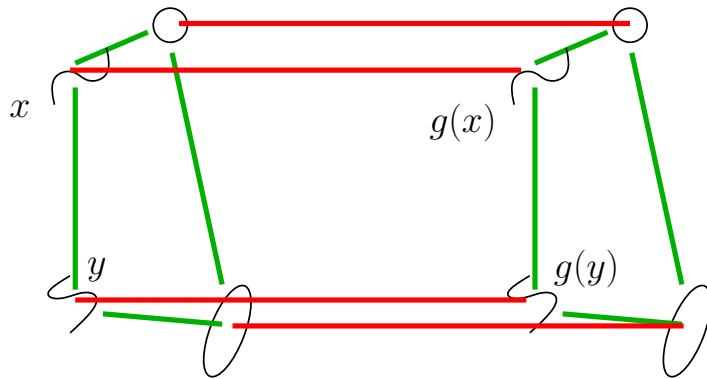


3D strategies—pair types



Type I pairs = local pairs or n -tuples

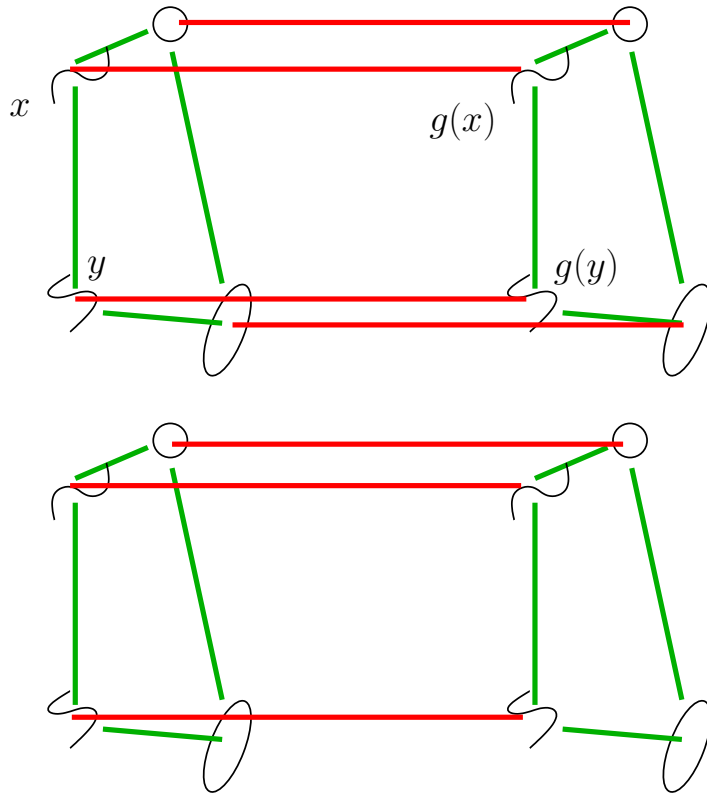
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Type II pairs = fundamental length pairs

3D strategies—pair types

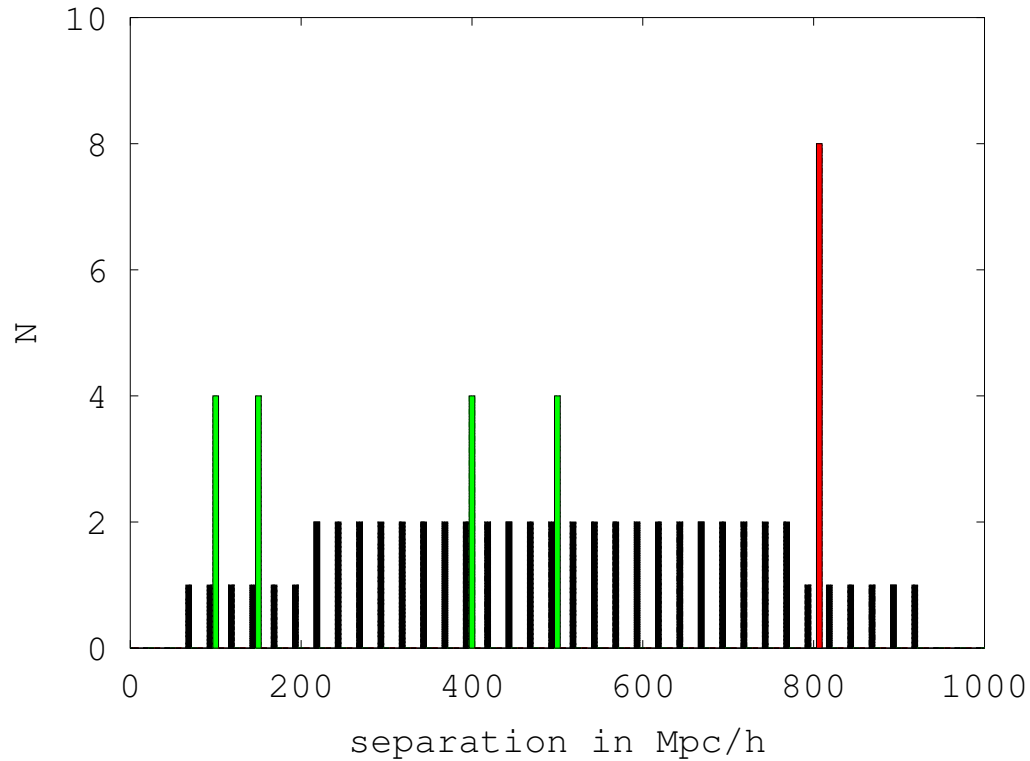


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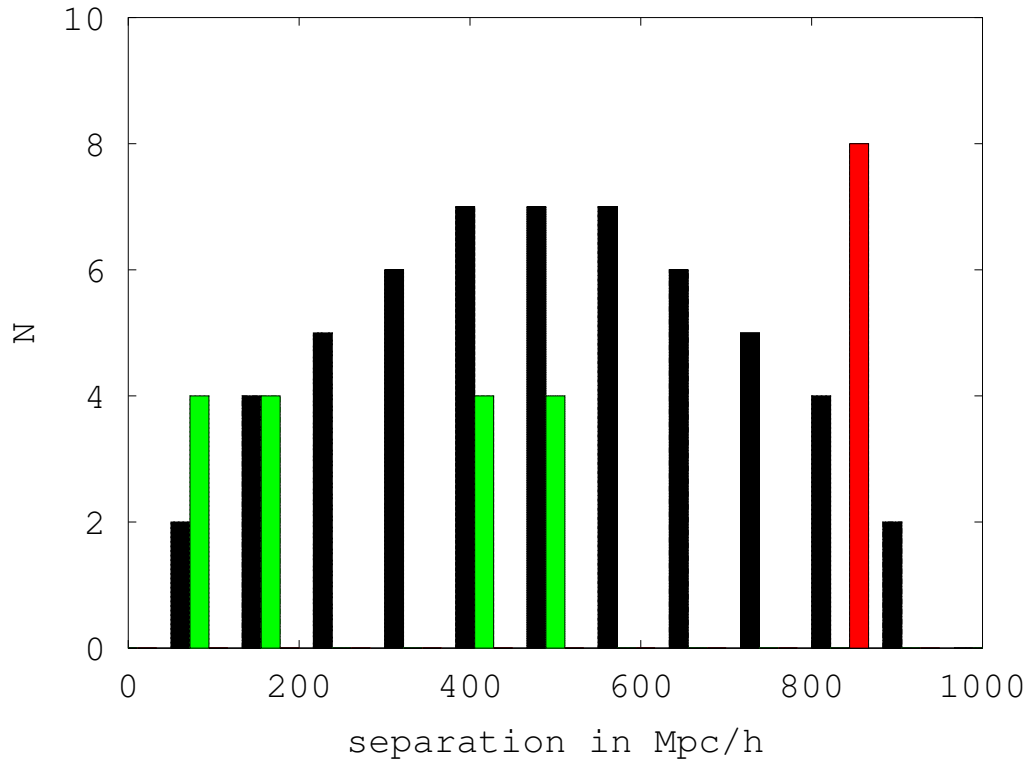


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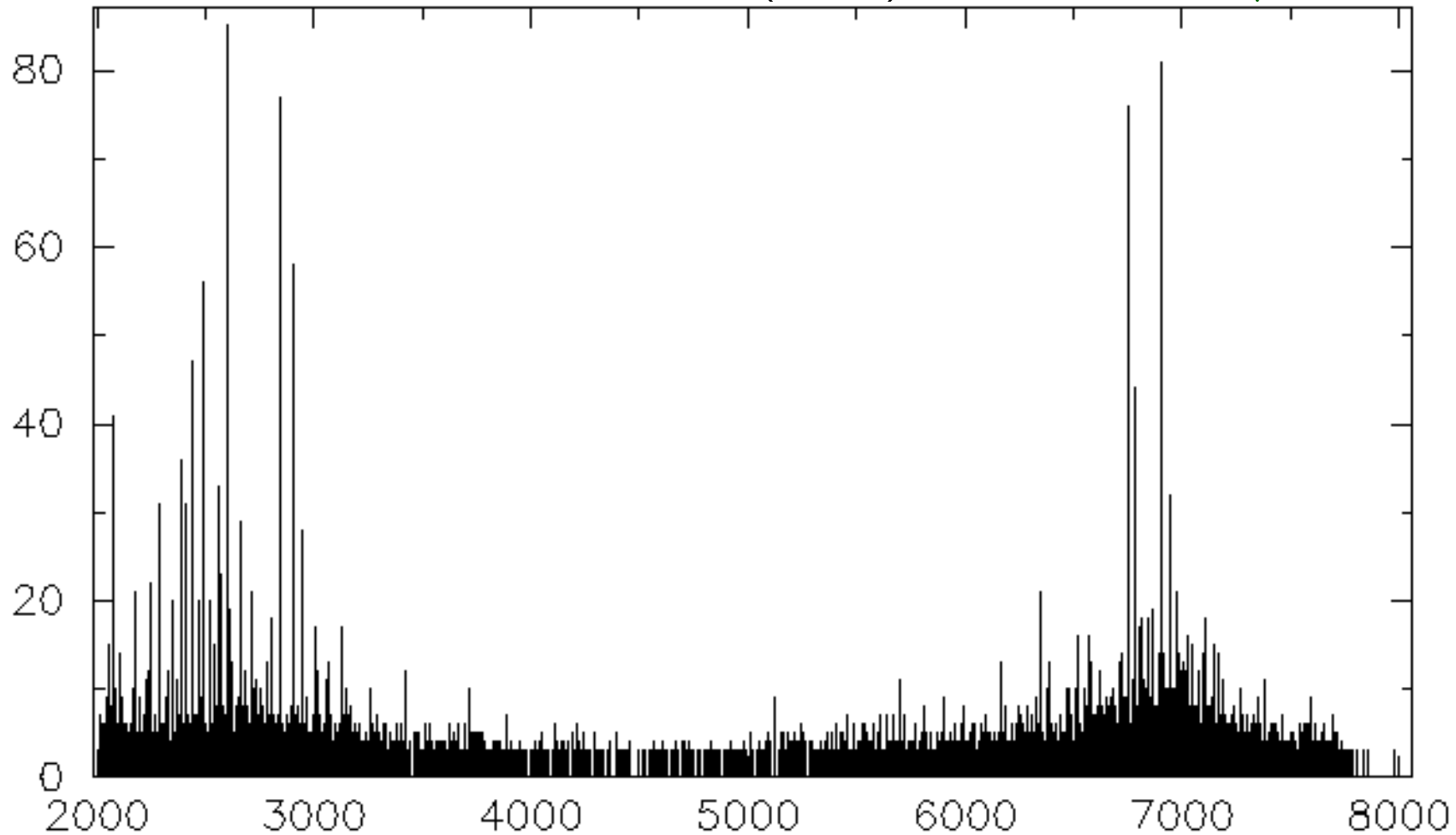
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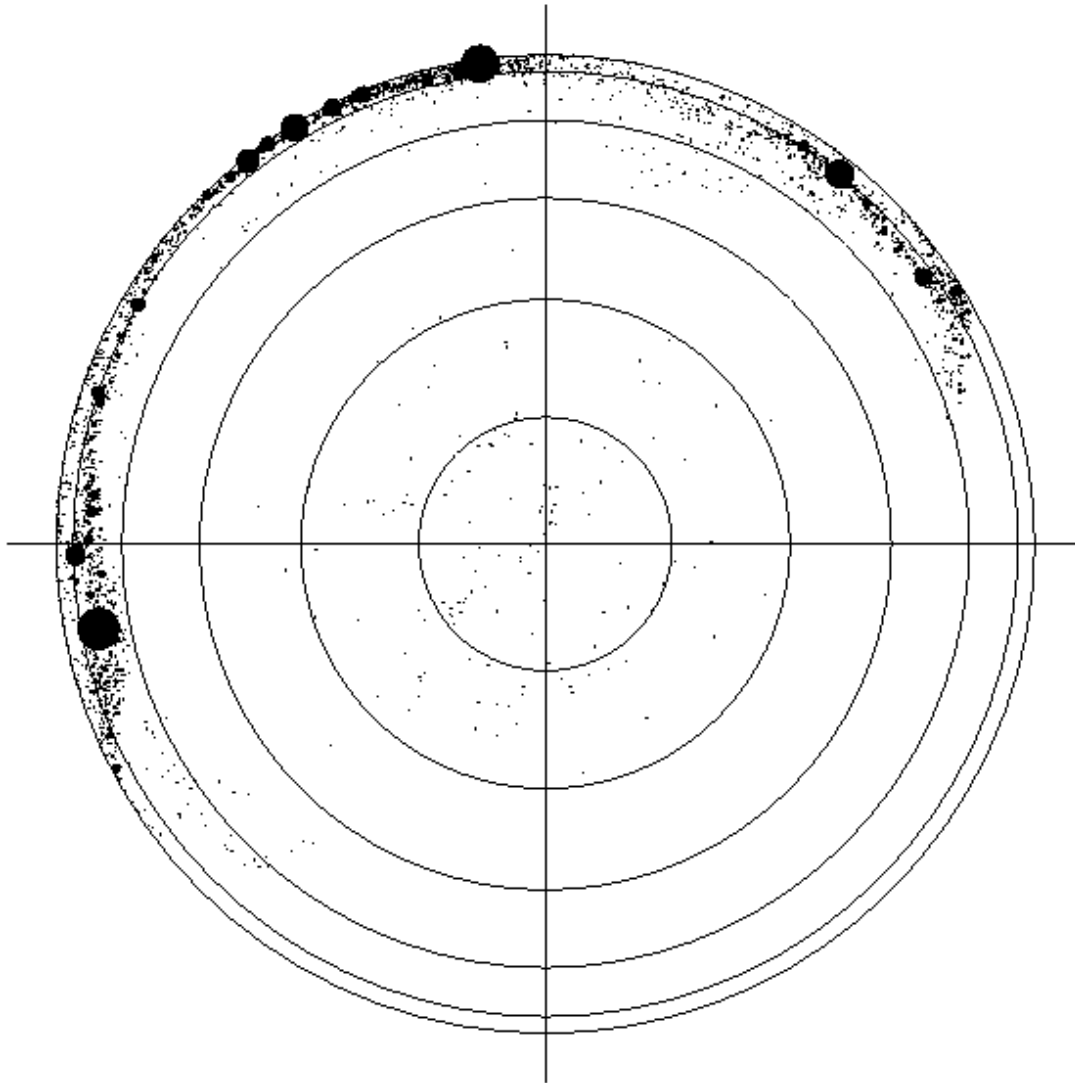
AGN Catalogues

Marecki, Roukema, Bajtlik (2005) [arXiv:astro-ph/0412181](https://arxiv.org/abs/astro-ph/0412181)



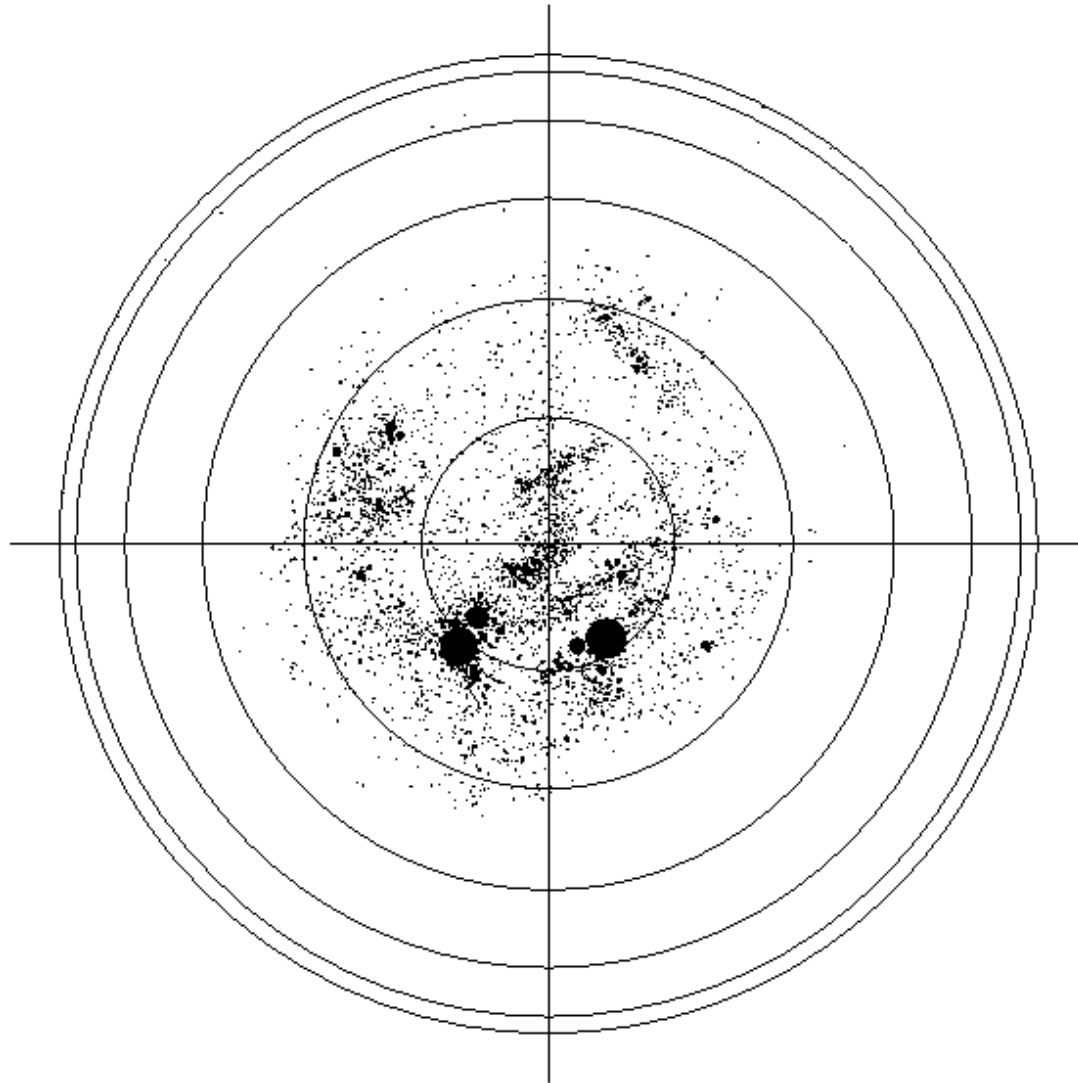
AGN Catalogues

gmod range: 2000 – 5000
ztol=0.50%, gtol=1.00%, angtol=0.0050 rad
gtolfact=100., angtolfact= 30., gmodmin= 50.
input file: analysepairs.qso_results, Omega_m=0.30



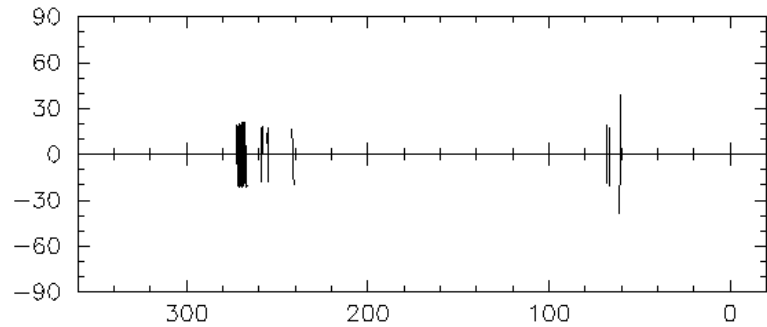
AGN Catalogues

gmod range: 5000 – 8000
ztol=0.50%, gtol=1.00%, angtol=0.0050 rad
gtolfact=100., angtolfact= 30., gmodmin= 50.
input file: analysepairs.qso_results, Omega_m=0.30

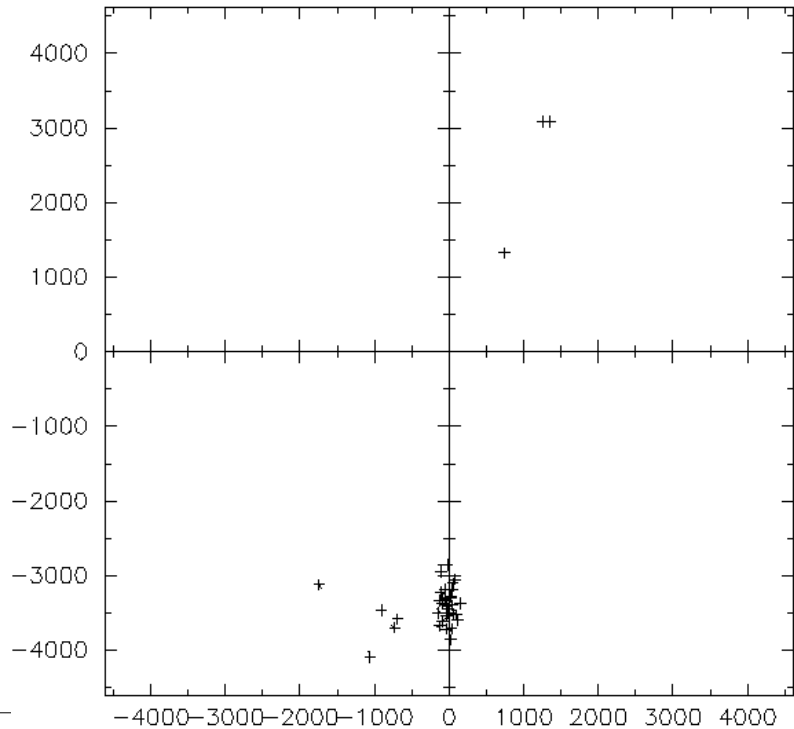


AGN Catalogues

Positions of objects on matched discs (weighted cleaned data)



RA=17 47 00.0 Dec=10 31 54 $l=35.253$ $b=19.0203$
group # 2265, number of pairs= 36, $g_{\text{mod}}=2387.1314$ Mpc



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2D strategies

- TODO...