



Special and General Relativity

Boud Roukema

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SR+GR



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- SR: spacetime = w:Minkowski space

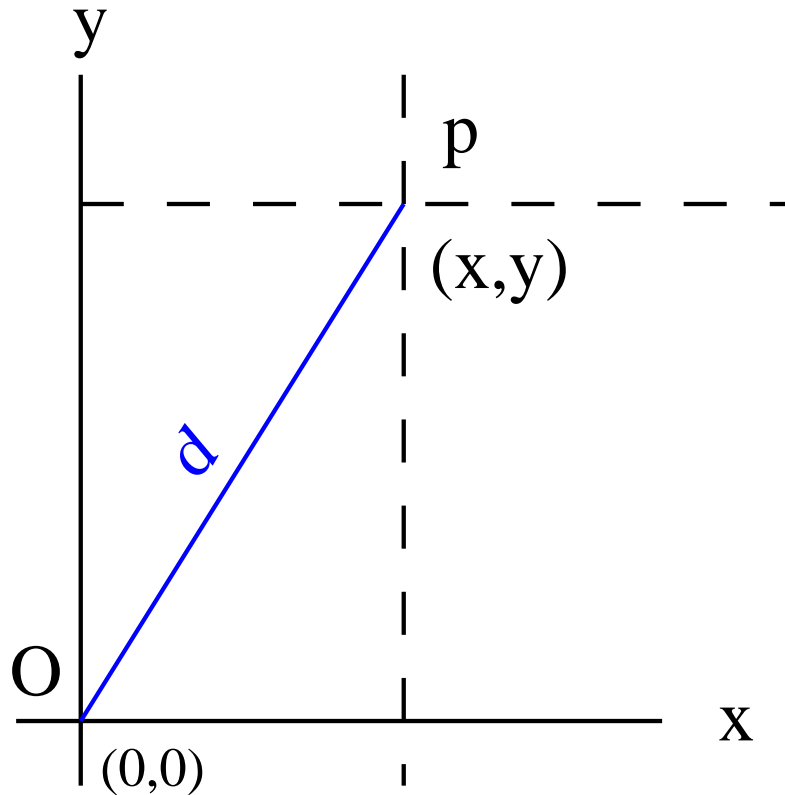


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- GR: spacetime = a solution of the w:Einstein field equations



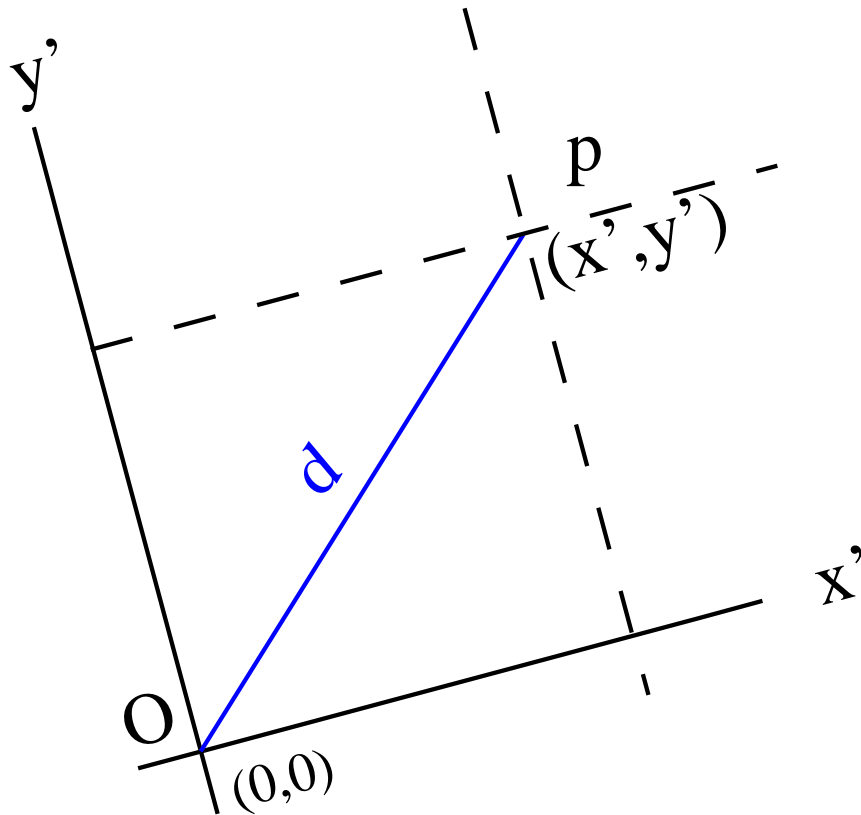
SR: Minkowski spacetime



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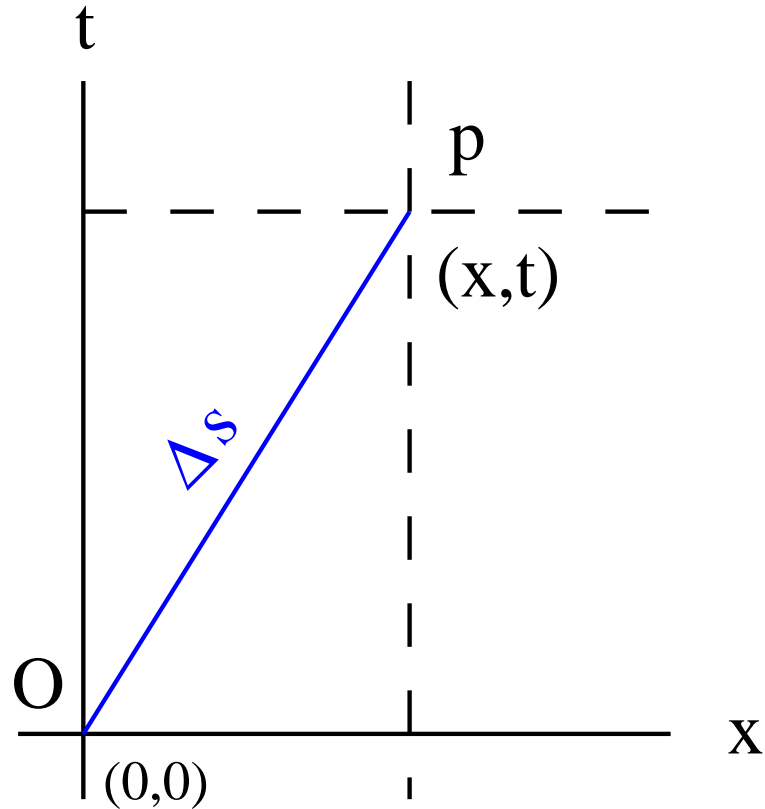
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$$\begin{pmatrix} p_{x'} \\ p_{y'} \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

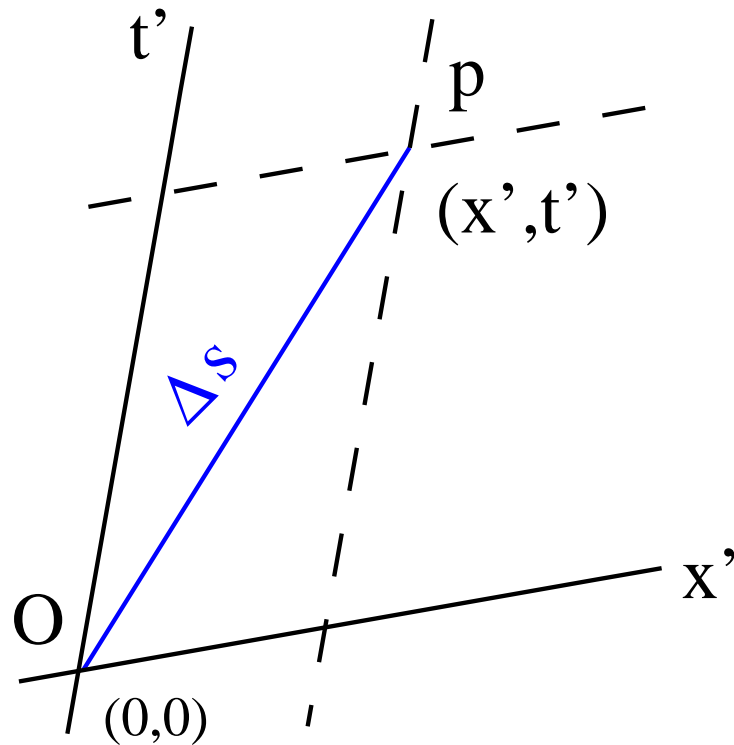
p at (x', y') , distance from observer at O is $d = \text{unchanged}$

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p at (x, t) , w:invariant interval from observer at O is Δs
where $(\Delta s)^2 = -(\Delta t)^2 + (\Delta x)^2$

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$$\begin{pmatrix} p_{x'} \\ p_{t'} \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix} \begin{pmatrix} p_x \\ p_t \end{pmatrix}$$

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SR: Lorentz transformation

$$\Lambda(\phi) := \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix}$$

$$\cosh \phi := \frac{e^\phi + e^{-\phi}}{2}$$

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w:hyperbolic function

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 $= \frac{1 \text{ s}}{1 \text{ s}} = 1$ (dimensionless)

SR: rapidity ϕ vs velocity β

What is ϕ ?



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observer A has worldline $(x, t) = (0, t)$



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in A's coordinate system, B's worldline is:

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$$\Rightarrow x = t' \sinh \phi = t \tanh \phi = \beta t$$

where velocity $\beta := v/c \equiv v = \tanh \phi$

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Where does $(x', t') = (0, 1)$ lie for observer A?



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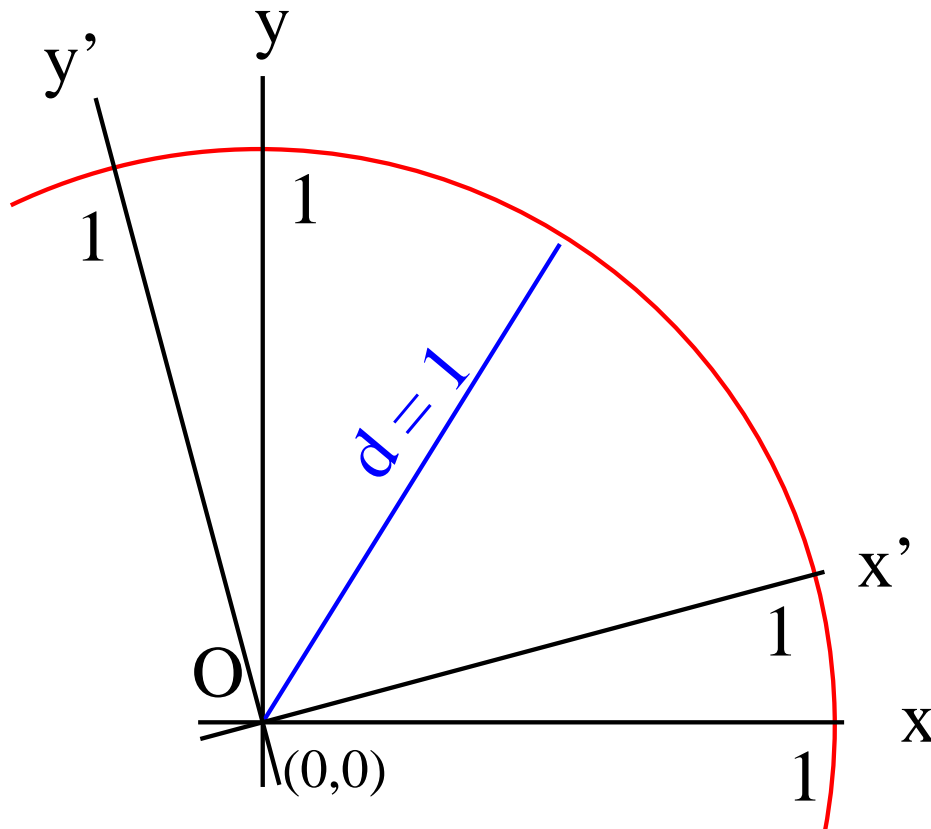
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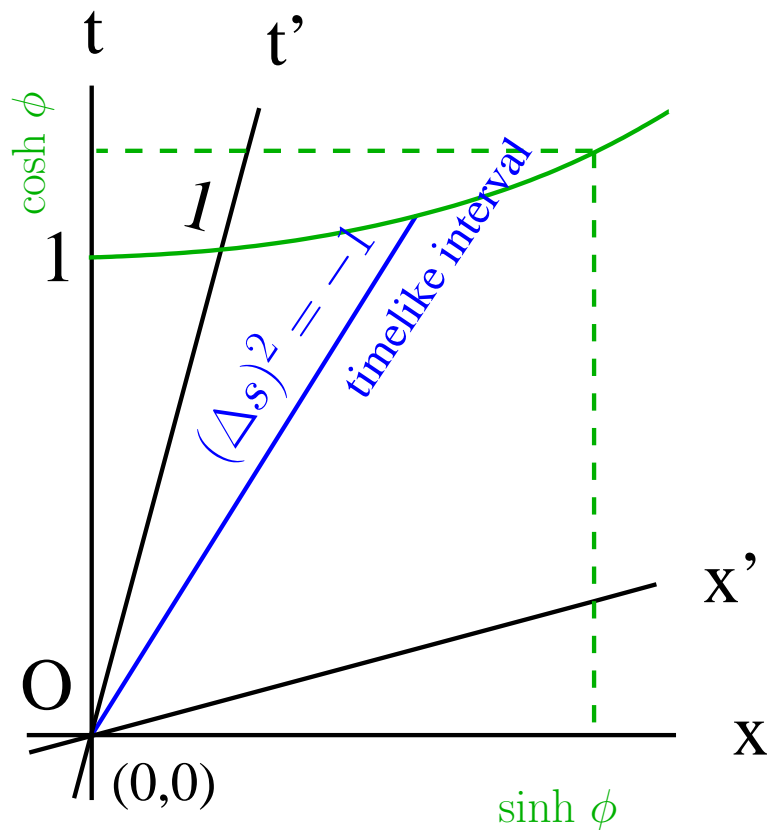


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 $d = 1$ from $(0, 0)$

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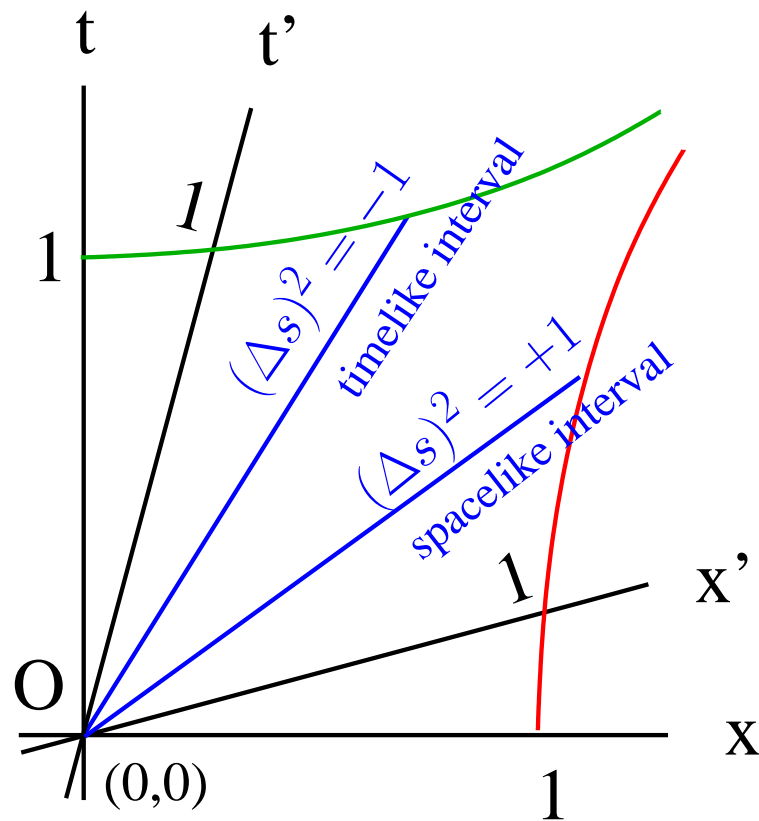
constant interval
 $(\Delta s)^2 = -1$ from $(0, 0)$

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Can high ϕ push the t' axis close to the x axis?

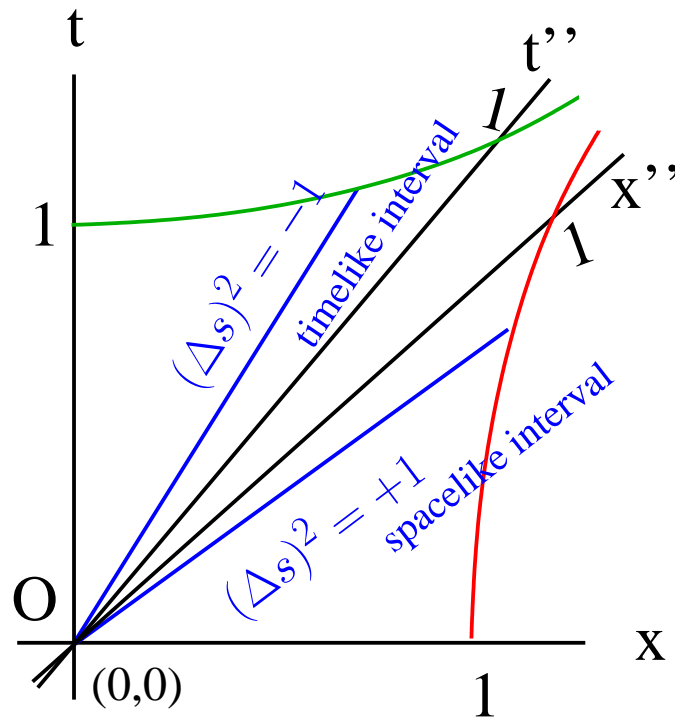


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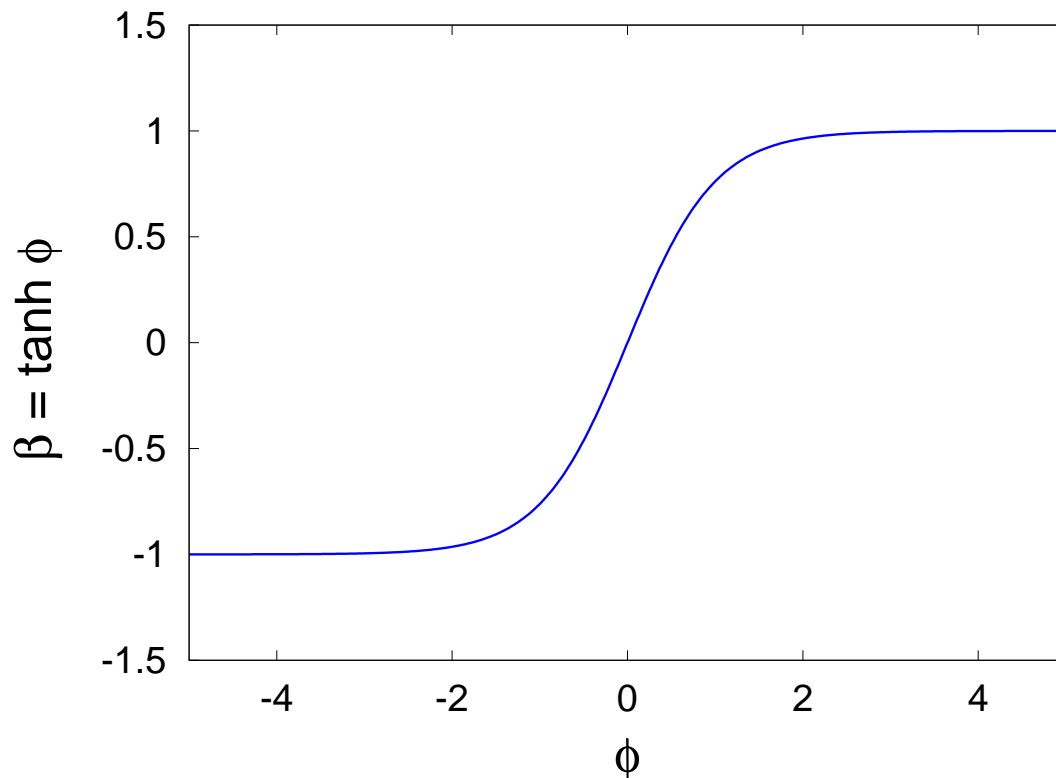


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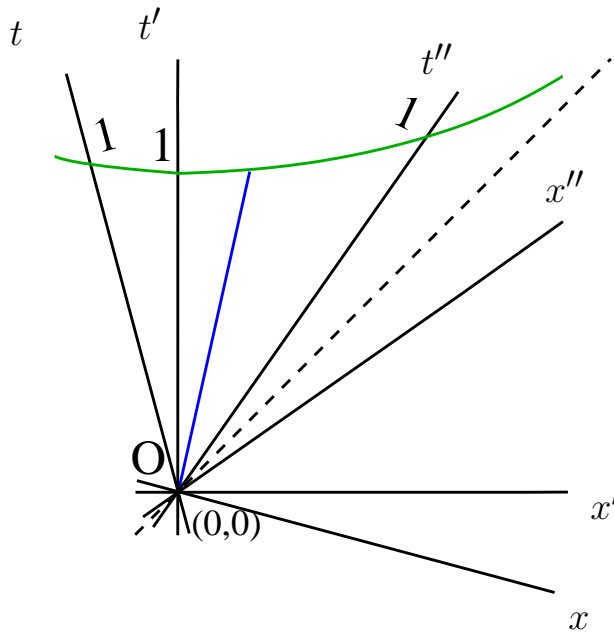
- photon speed same in both reference frames
- w:Michelson-Morley experiment (1887)

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?

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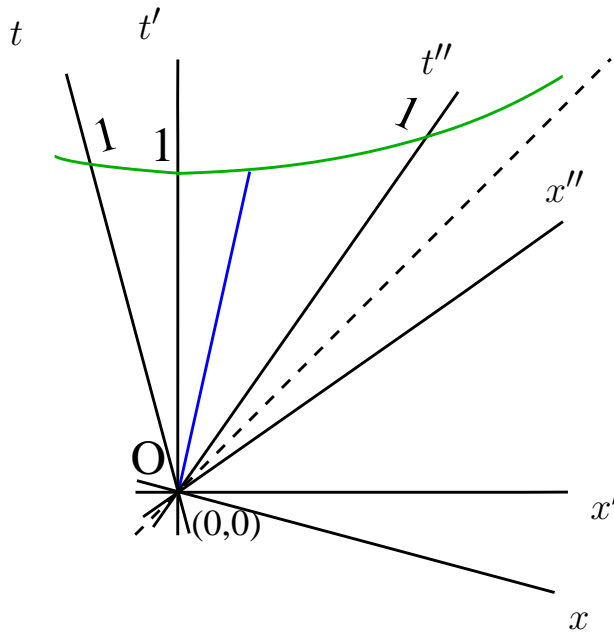


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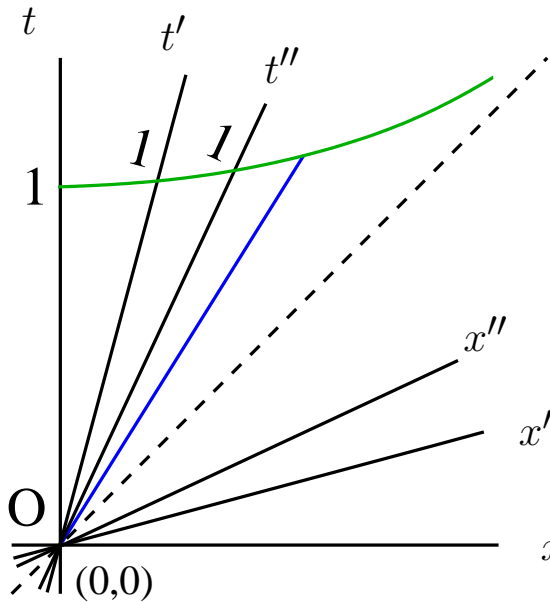


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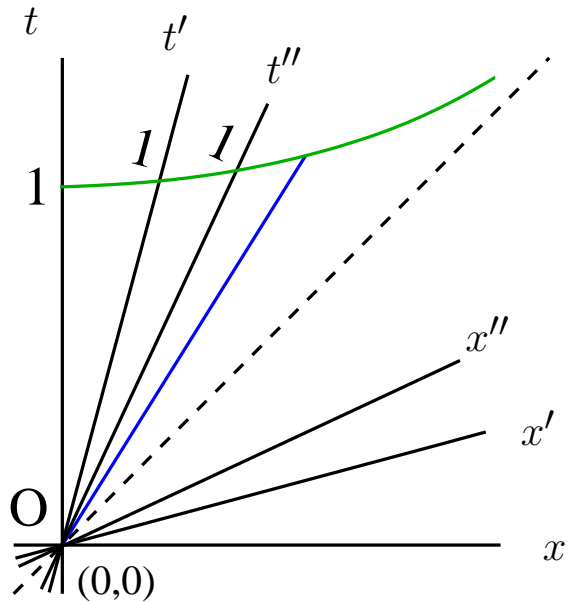
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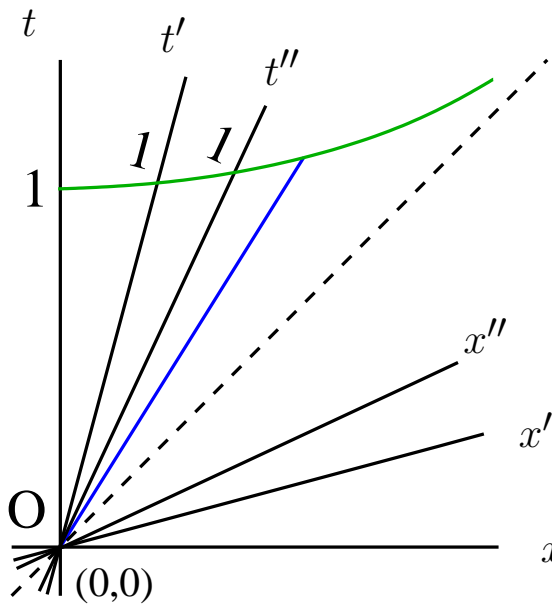
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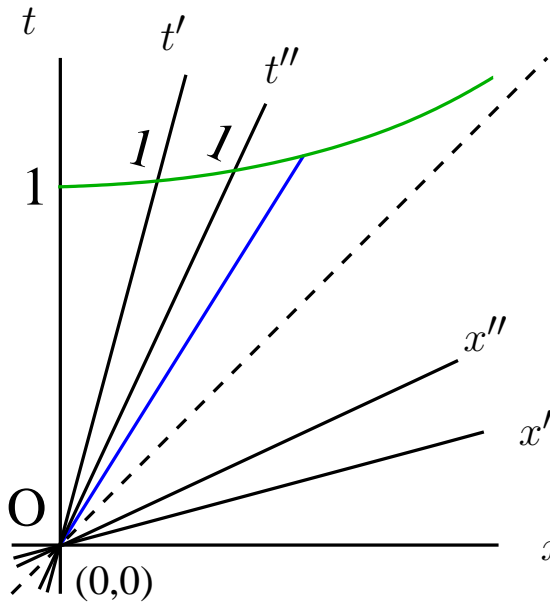


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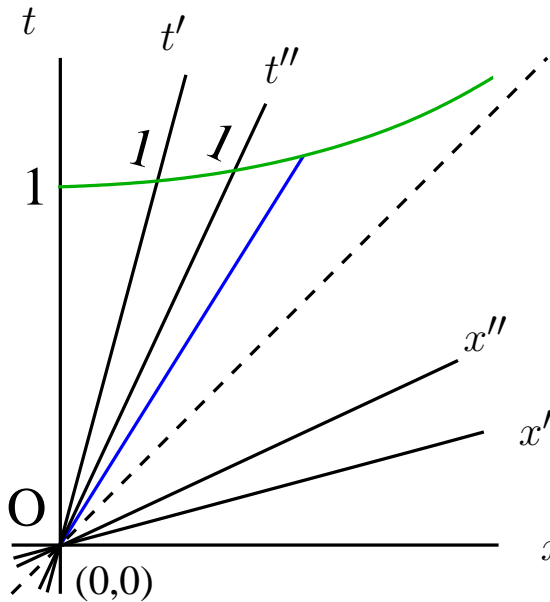
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cf. rotation θ_1 "plus" rotation $\theta_2 =$ rotation $\theta_1 + \theta_2$

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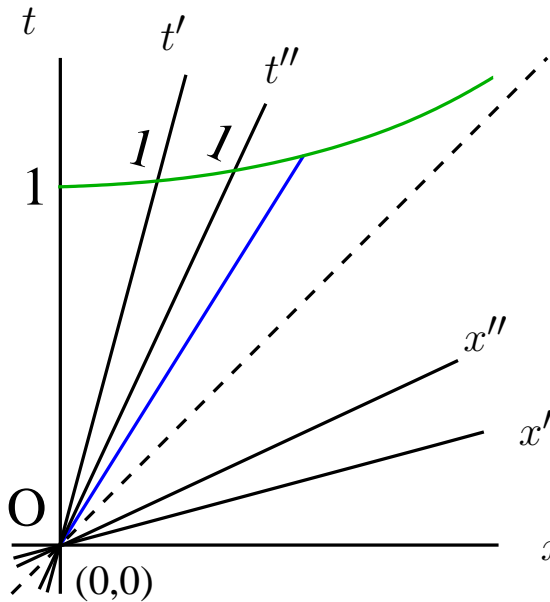
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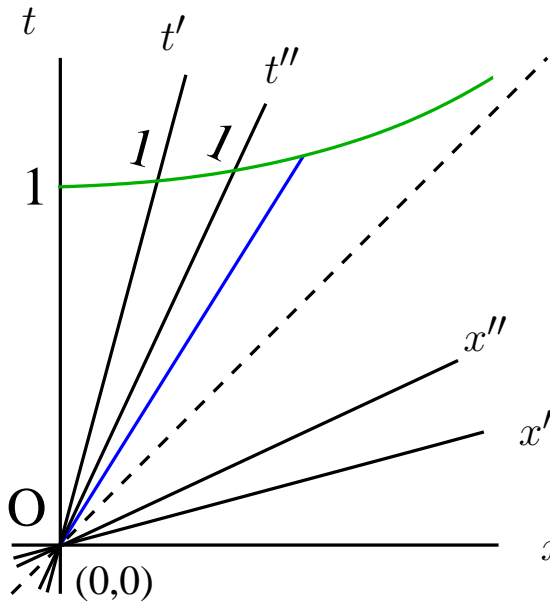
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so $\beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2}$

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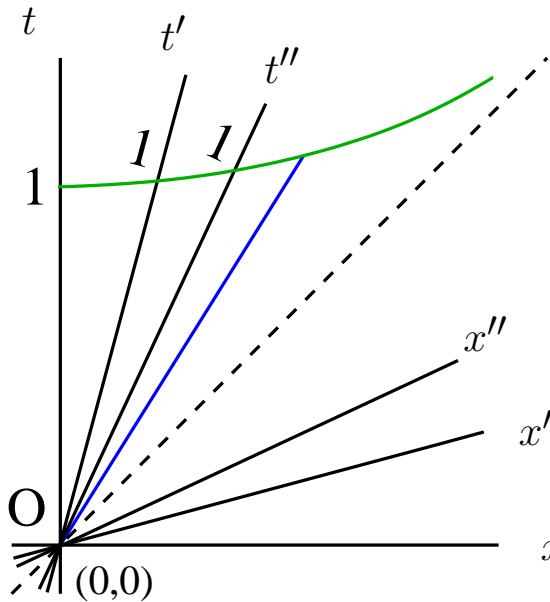
$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

so $\beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2}$

SR: adding velocities

interstellar ark travels at $\beta_1 = 0.1$ from Sun, sends out rocket at $\beta_2 = 0.5$; rocket's speed β_3 in Sun frame =?



$$\begin{pmatrix} x'' \\ t'' \end{pmatrix} = \Lambda(\phi_2)\Lambda(\phi_1) \begin{pmatrix} x \\ t \end{pmatrix}$$

but $\Lambda(\phi_2)\Lambda(\phi_1) = \Lambda(\phi_1 + \phi_2)$

$$\text{so } \beta_3 = \tanh(\phi_1 + \phi_2) = \frac{\beta_1 + \beta_2}{1 + \beta_1\beta_2} = \frac{0.1 + 0.5}{1 + 0.1 \times 0.5} \approx 0.57$$

SR: Lorentz factor



Λ : alternative to hyperbolic trig functions



SR: Lorentz factor



Λ : alternative to hyperbolic trig functions

$$\Lambda(\phi) := \begin{pmatrix} \cosh \phi & -\sinh \phi \\ -\sinh \phi & \cosh \phi \end{pmatrix}$$

$$\cosh \phi := \frac{e^\phi + e^{-\phi}}{2}$$

$$\sinh \phi := \frac{e^\phi - e^{-\phi}}{2}$$

w:hyperbolic function



SR: Lorentz factor



Λ : alternative to hyperbolic trig functions

$$\Lambda(\beta) := \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$$

$$\beta = \tanh \phi$$

$$\gamma := (1 - \beta^2)^{-1/2} =$$

Lorentz factor

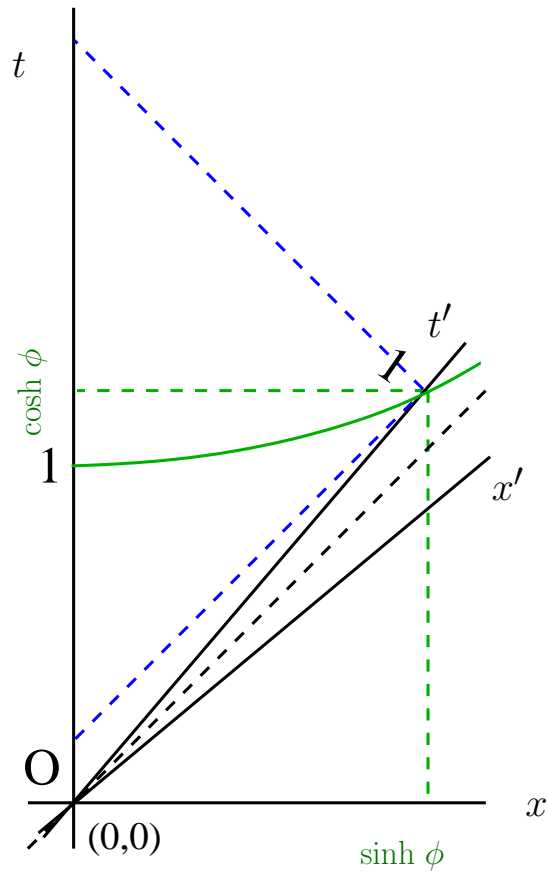
$$\gamma = \cosh \phi$$

$$\beta\gamma = \sinh \phi$$



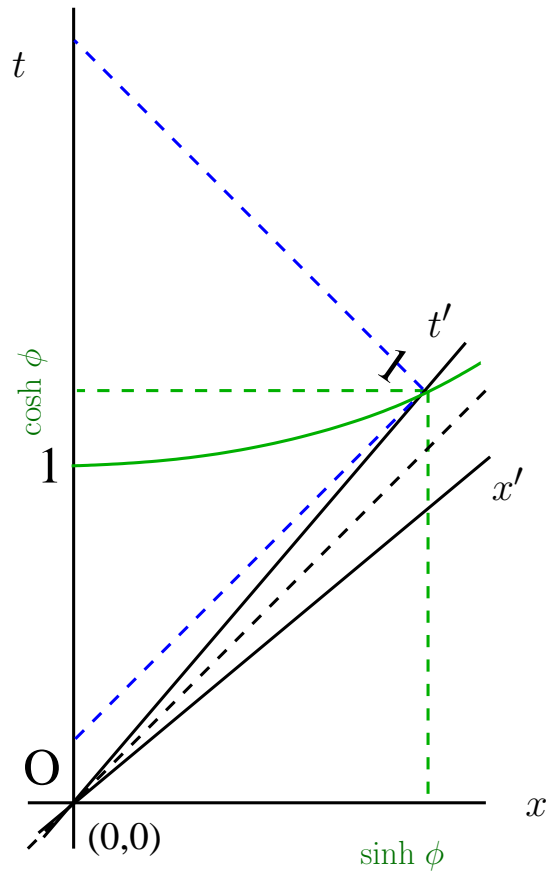


SR: worldline time dilation





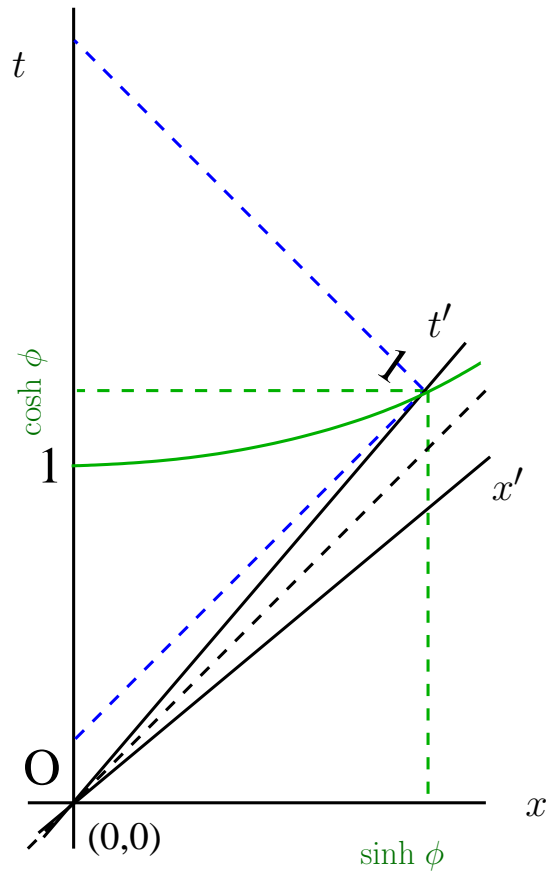
SR: worldline time dilation



$$\cosh \phi \equiv \gamma$$



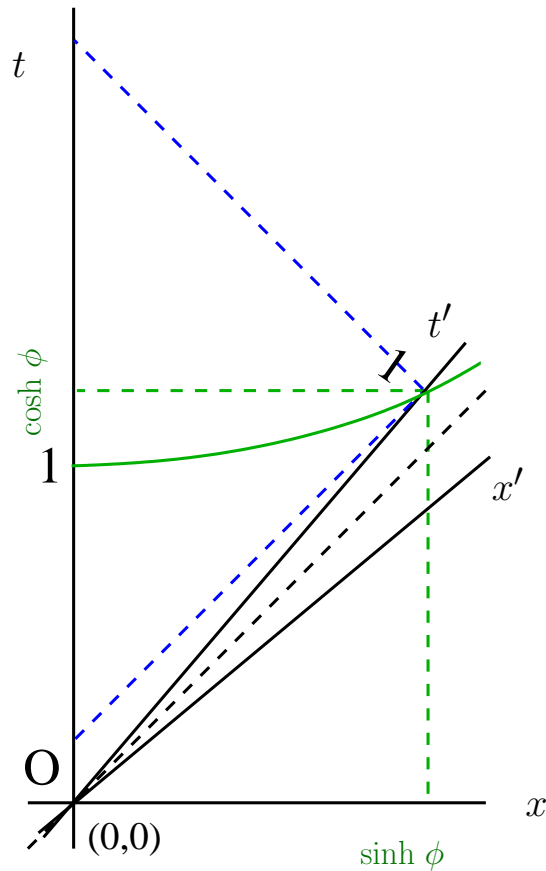
SR: worldline time dilation



$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$



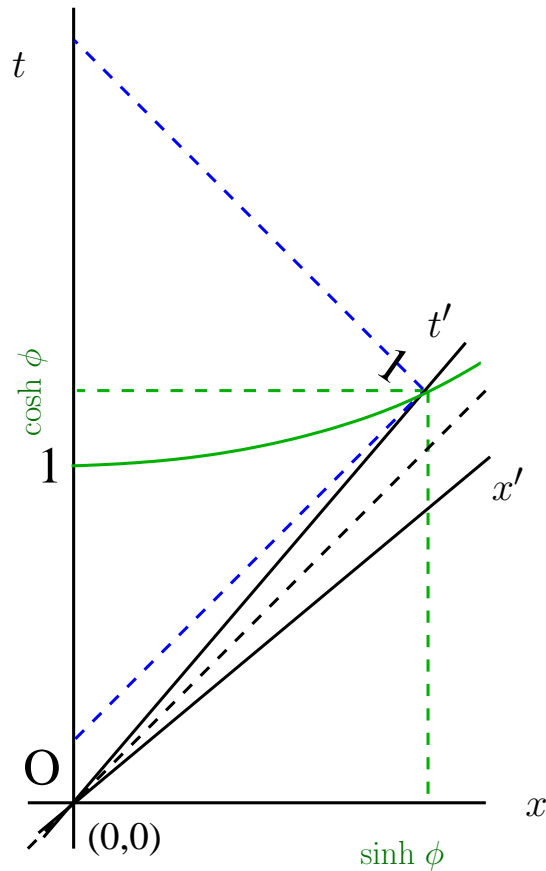
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$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$



SR: worldline time dilation

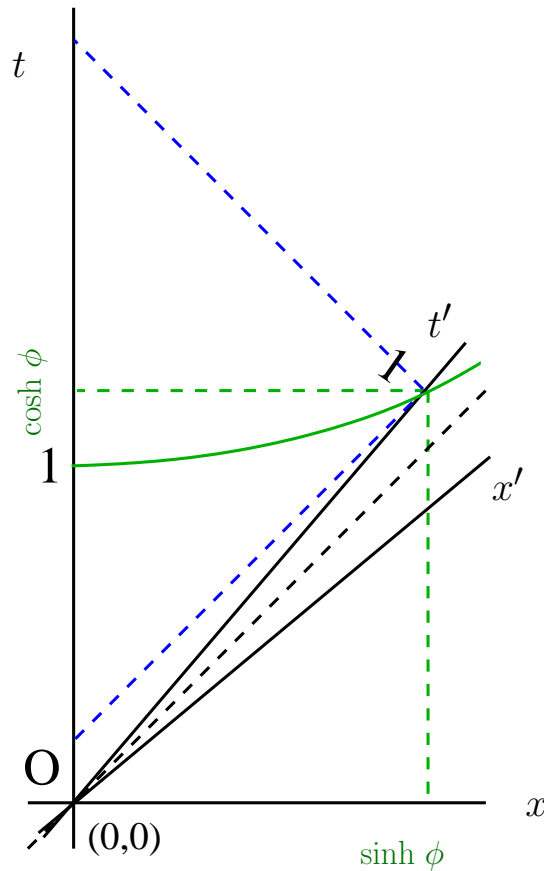


worldline “time dilation”

$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$



SR: worldline time dilation



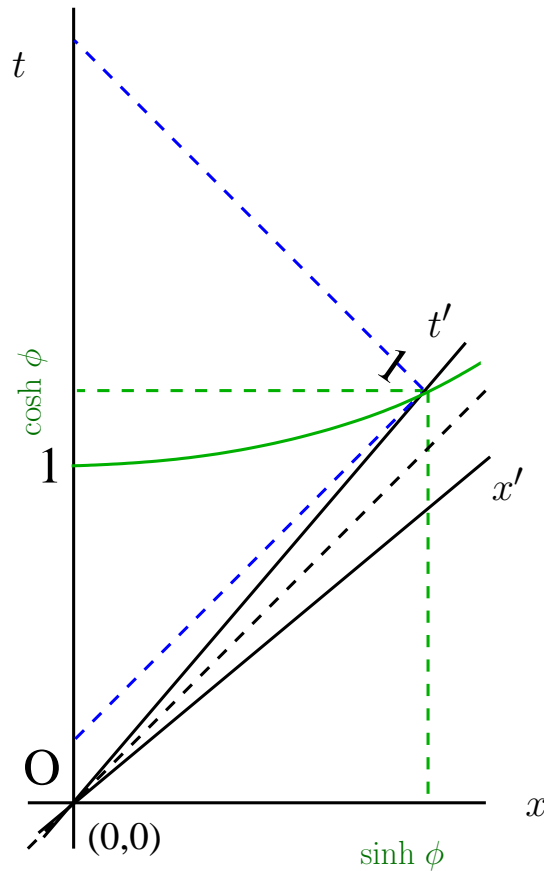
worldline “time dilation”

muons: mean lifetime
2197 ns \ll 15 km

$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$



SR: worldline time dilation



worldline “time dilation”

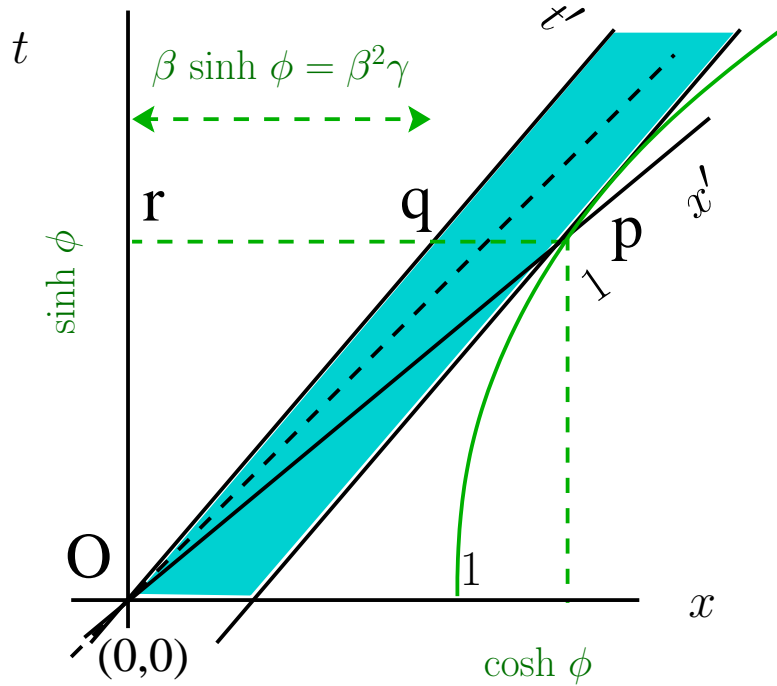
muons: mean lifetime
2197 ns \ll 15 km

time dilation \Rightarrow muons
can hit the ground

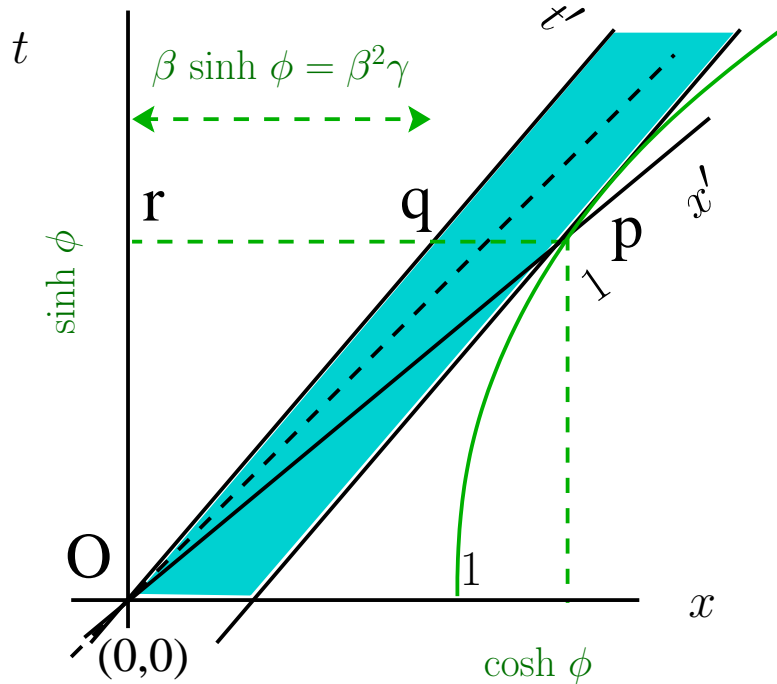
$$\cosh \phi \equiv \gamma \equiv \frac{1}{\sqrt{1-\beta^2}} > 1$$



SR: worldsheet space contraction



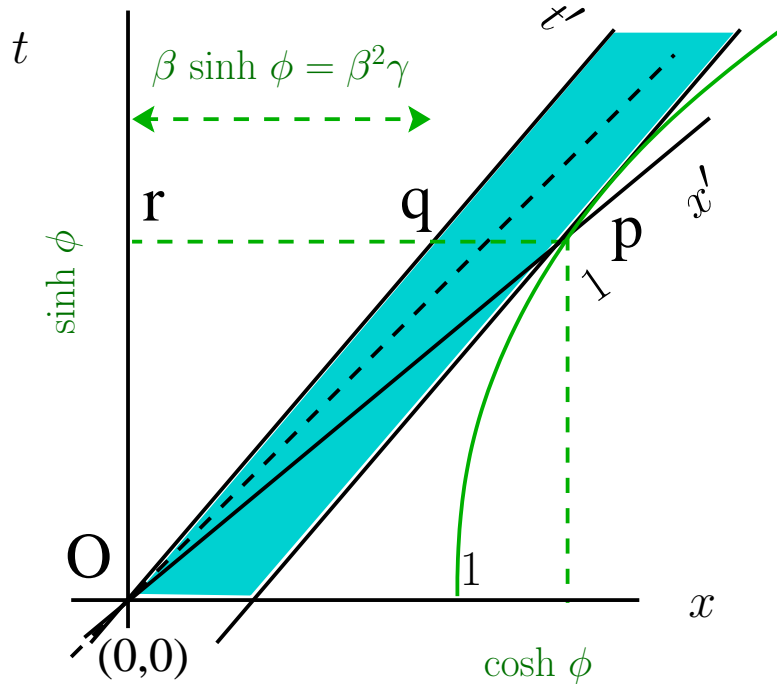
SR: worldsheet space contraction



$$\sqrt{\Delta s^2(q, p)} =$$

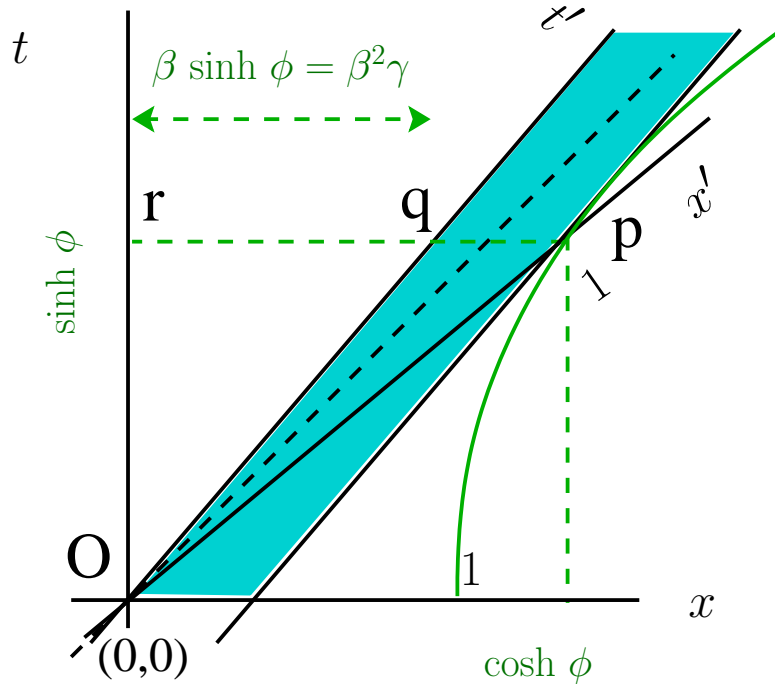


SR: worldsheet space contraction



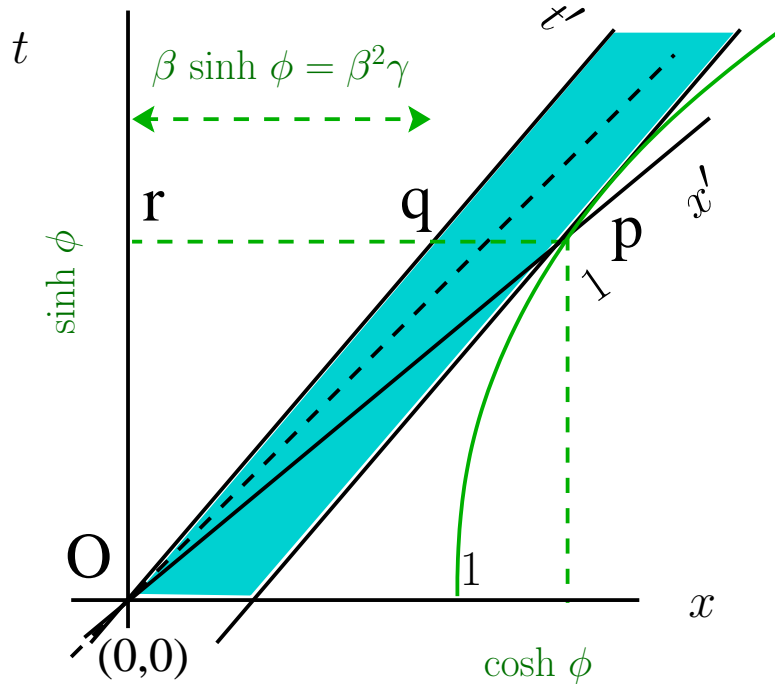
$$\sqrt{\Delta s^2(q, p)} = \cosh \phi - \beta \sinh \phi$$

SR: worldsheet space contraction



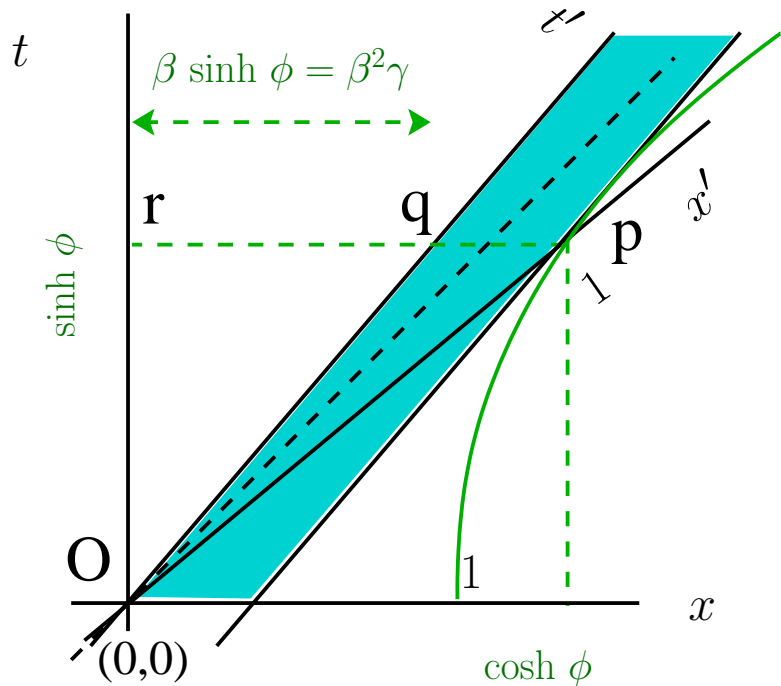
$$\sqrt{\Delta s^2(q, p)} = \gamma - \beta \beta \gamma$$

SR: worldsheet space contraction



$$\sqrt{\Delta s^2(q, p)} = (1 - \beta^2)\gamma$$

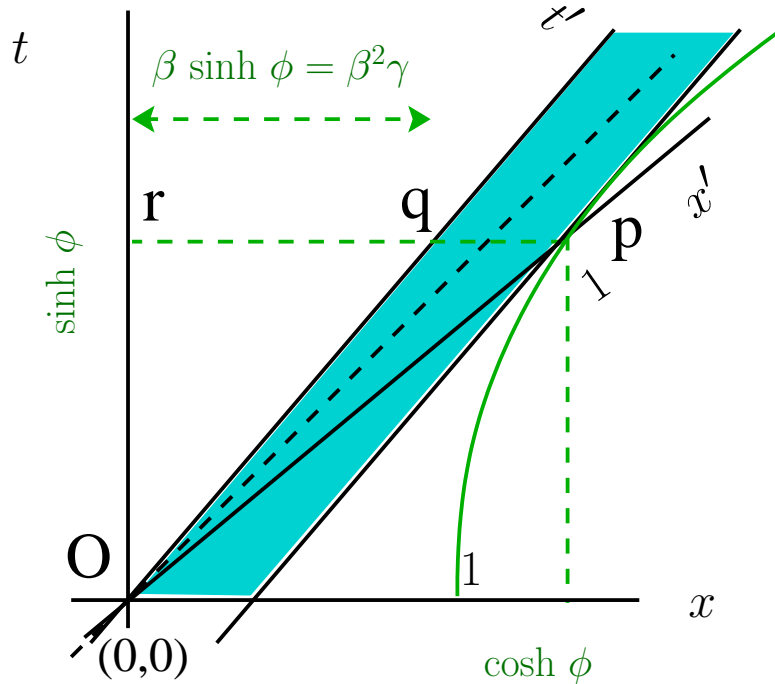
SR: worldsheet space contraction



$$\sqrt{\Delta s^2(q, p)} = (1 - \beta^2)^{1+(-1/2)}$$

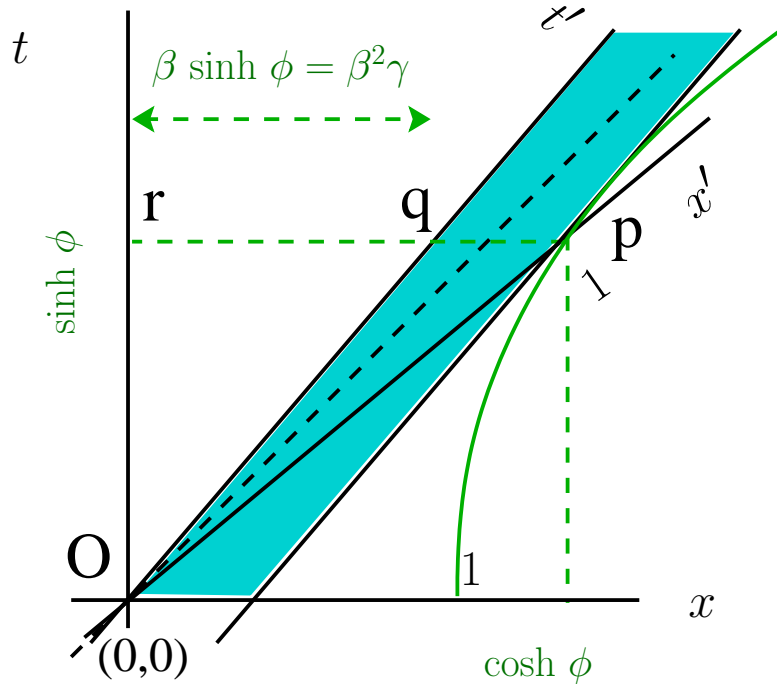


SR: worldsheet space contraction



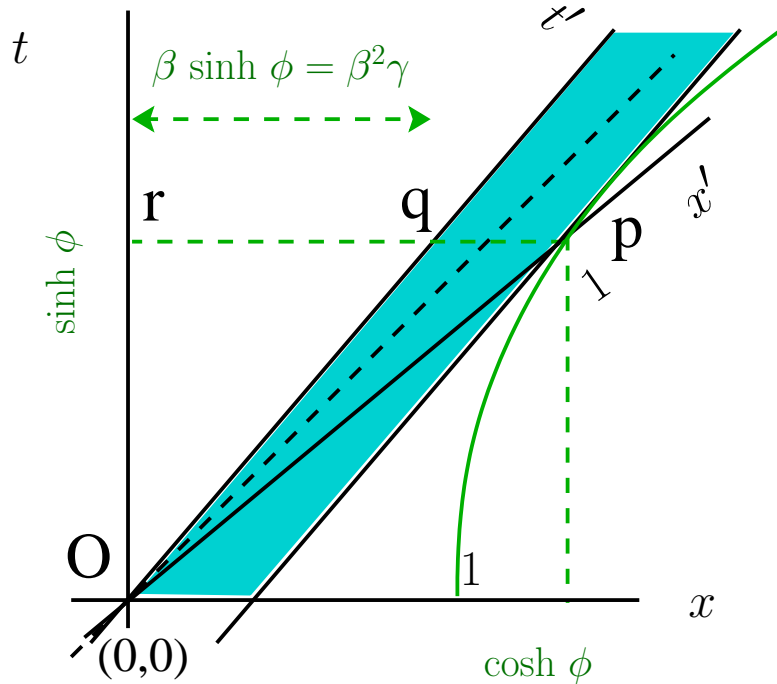
$$\sqrt{\Delta s^2(q, p)} = (1 - \beta^2)^{1/2}$$

SR: worldsheet space contraction



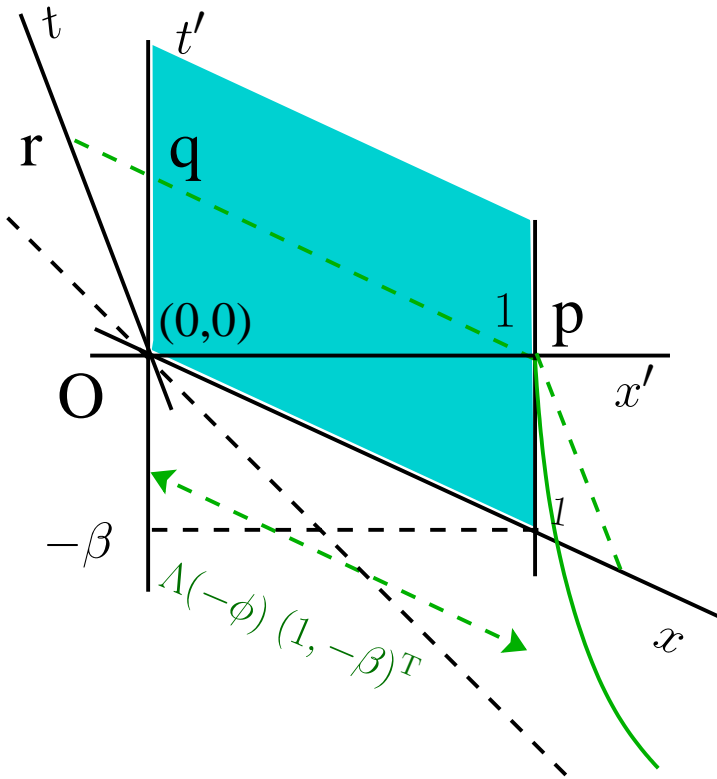
$$\sqrt{\Delta s^2(q, p)} = \gamma^{-1} < 1$$

SR: worldsheet space contraction

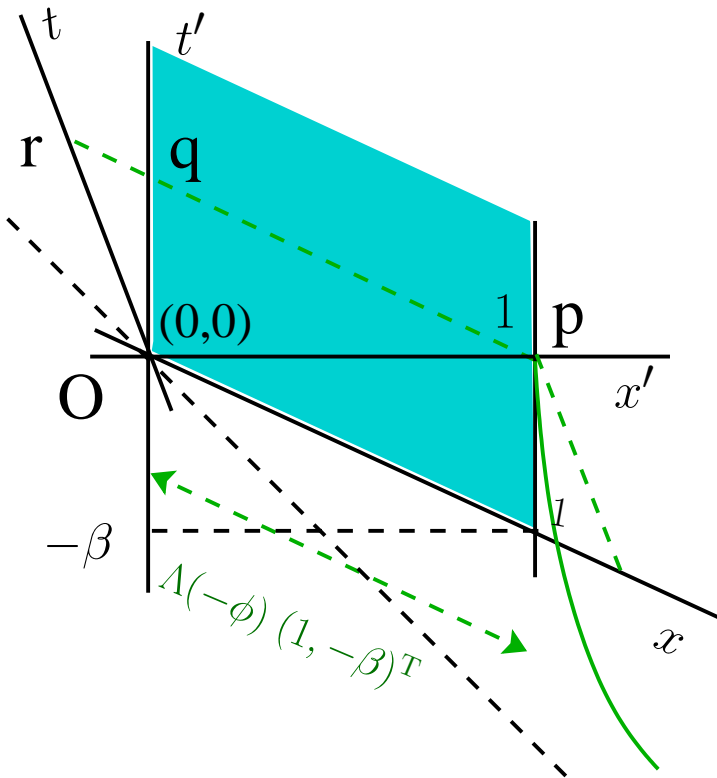


$$\sqrt{\Delta s^2(q, p)} = \gamma^{-1} < 1 \quad \text{worldsheet "space contraction"}$$

SR: worldsheet space contraction

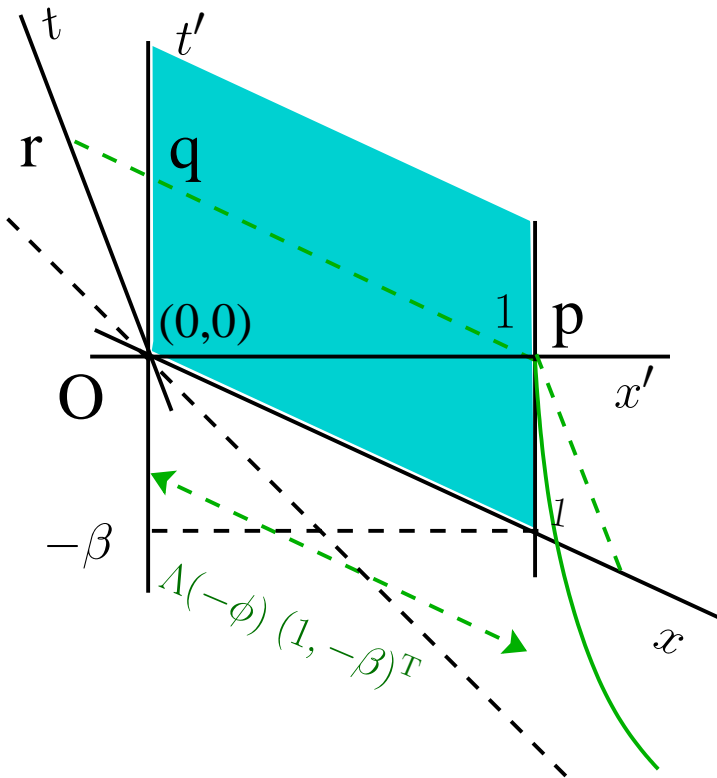


SR: worldsheet space contraction



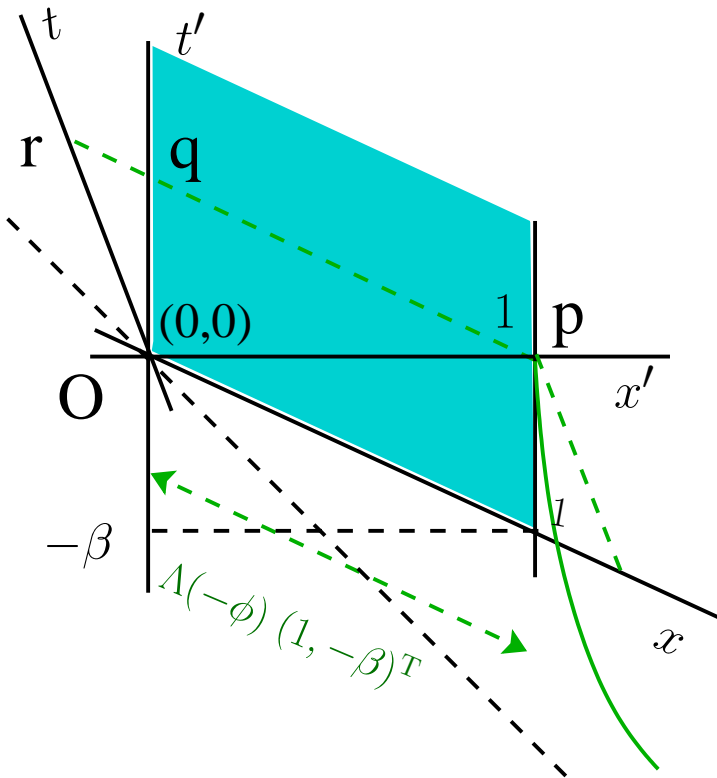
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \cosh \phi - \beta \sinh \phi \\ \sinh \phi - \beta \cosh \phi \end{pmatrix}$$

SR: worldsheet space contraction



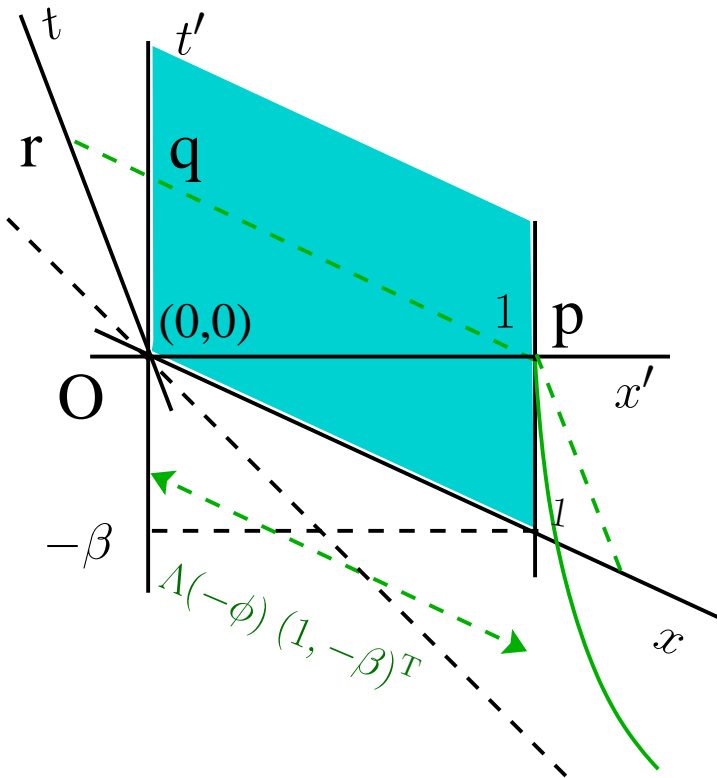
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \cosh \phi (1 - \beta^2) \\ \cosh \phi (\tanh \phi - \tanh \phi) \end{pmatrix}$$

SR: worldsheet space contraction



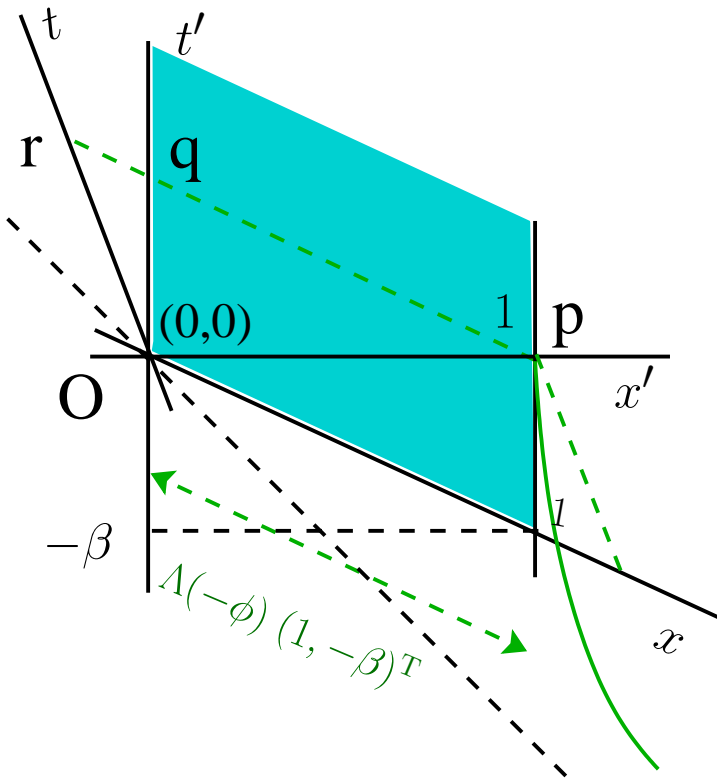
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \gamma(1 - \beta^2) \\ 0 \end{pmatrix}$$

SR: worldsheet space contraction



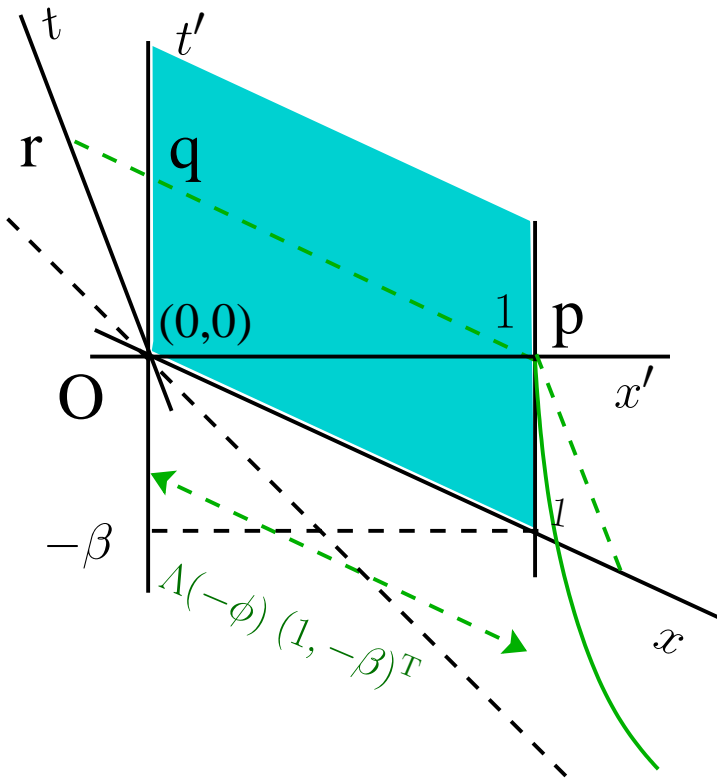
$$\Lambda^{-1} \begin{pmatrix} 1 \\ -\beta \end{pmatrix} = \begin{pmatrix} \gamma^{-1} \\ 0 \end{pmatrix}$$

SR: worldsheet space contraction



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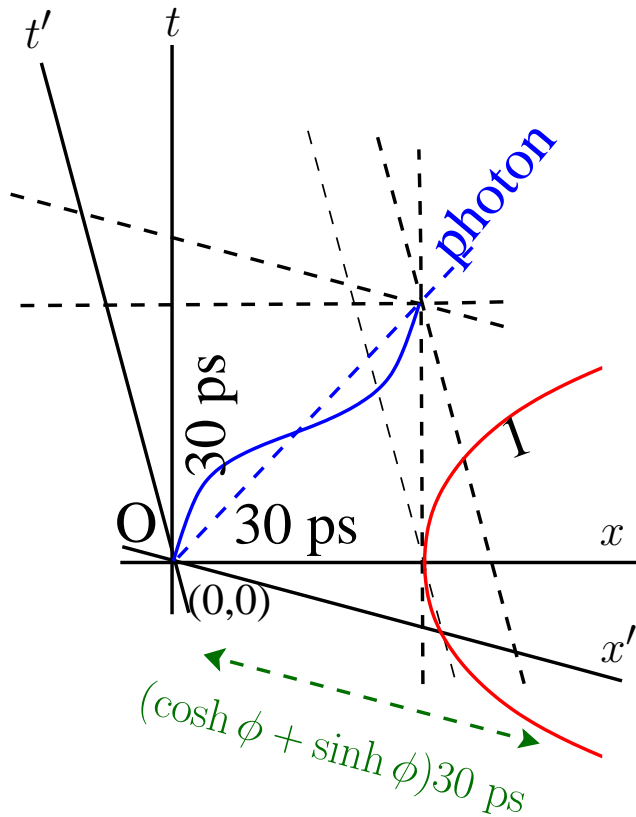
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$$\sqrt{\Delta s^2(q, p)} = \gamma^{-1} < 1 \quad \text{worldsheet "space contraction"}$$

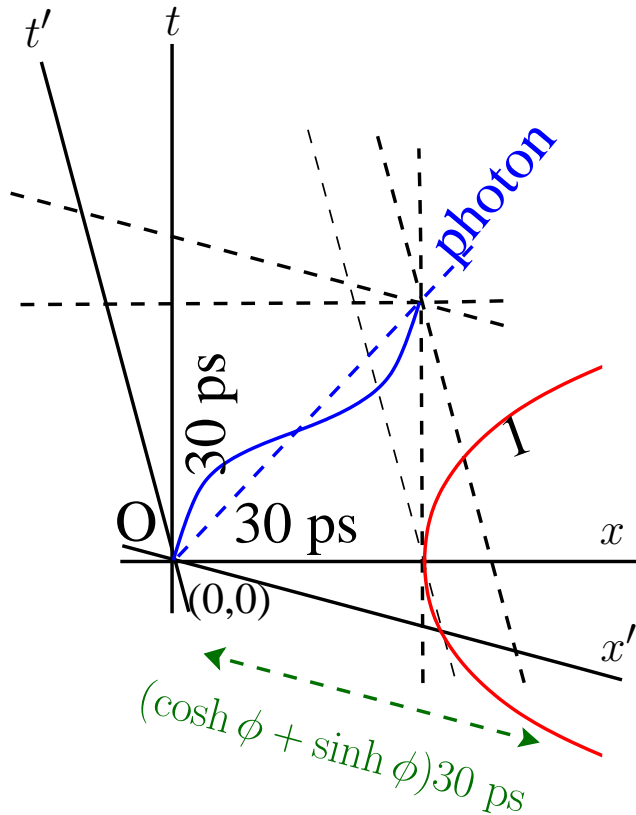


SR: Doppler shift





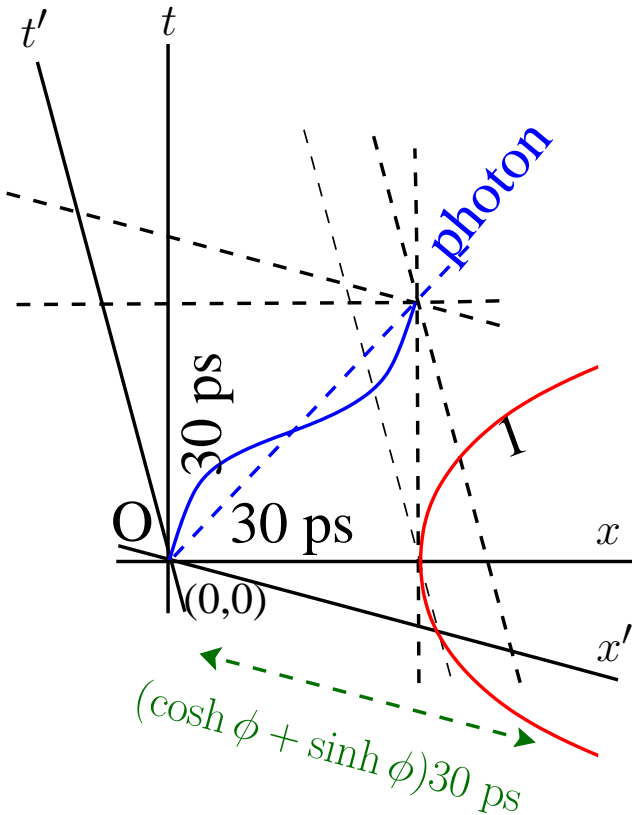
SR: Doppler shift



see photon worldline calculation



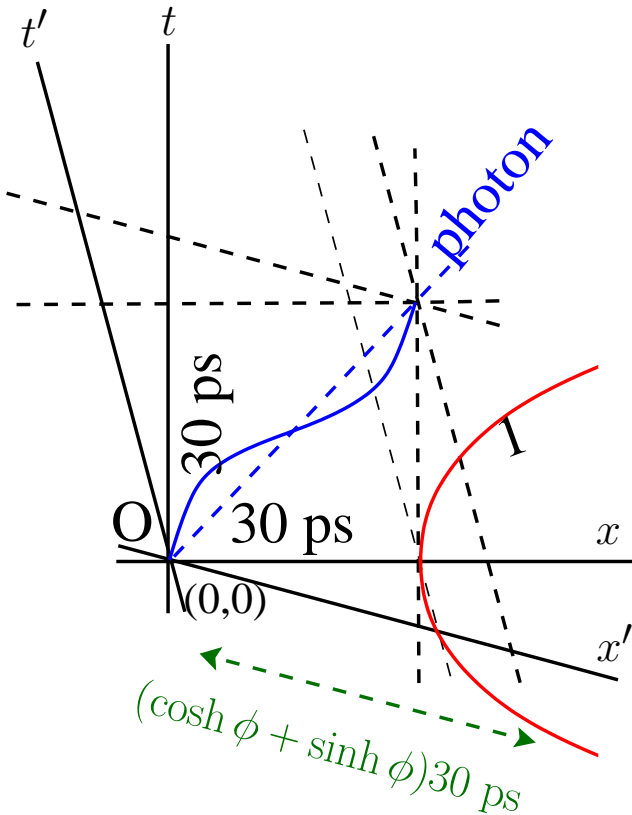
SR: Doppler shift



see photon worldline calculation

$$x' = (\cosh \phi + \sinh \phi)t$$

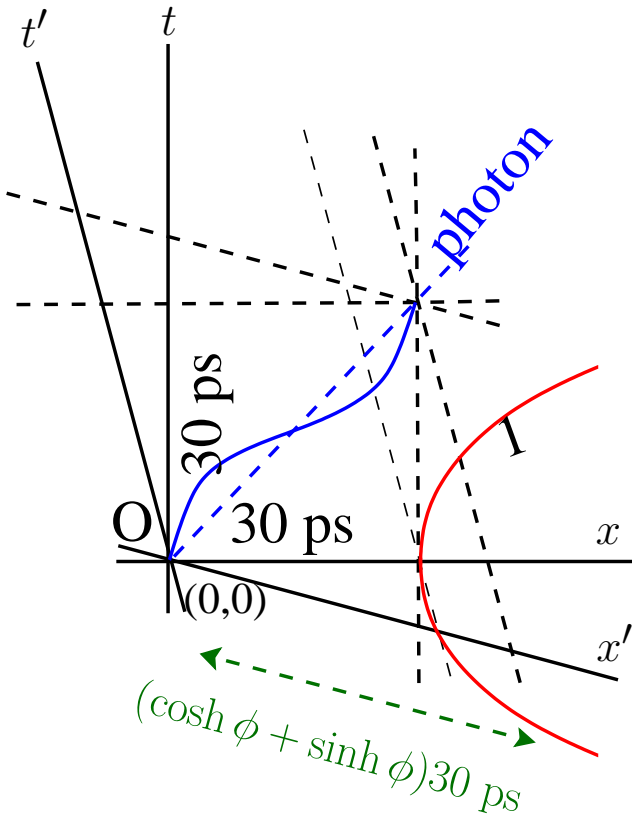
SR: Doppler shift



see photon worldline calculation

$$x' = (\cosh \phi + \sinh \phi)x$$

SR: Doppler shift

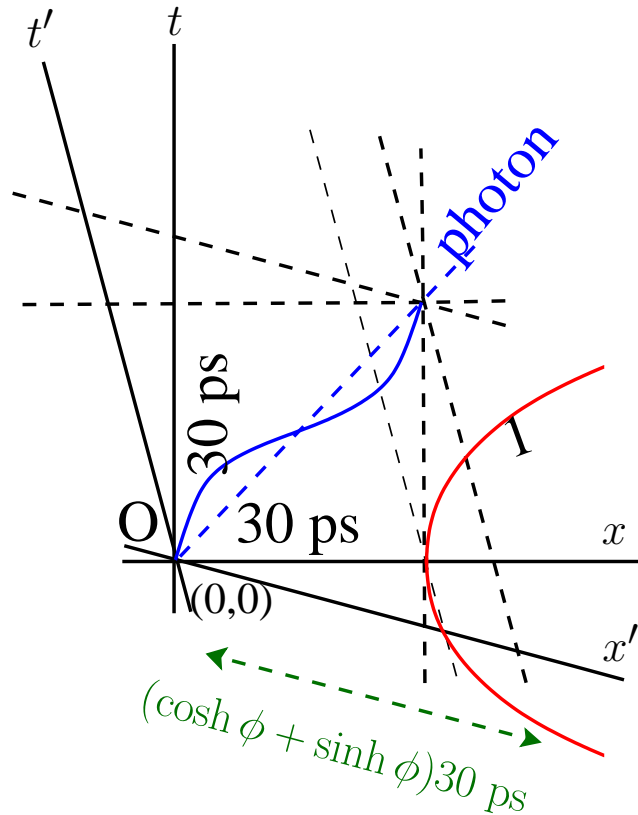


see photon worldline calculation

$$x'/x = \cosh \phi + \sinh \phi$$



SR: Doppler shift



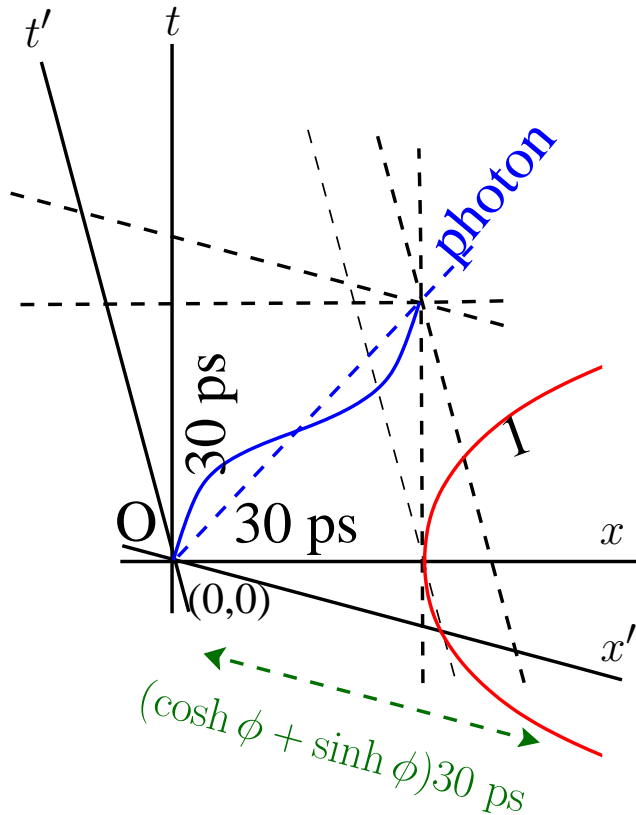
see photon worldline calculation

$$x'/x = \gamma + \beta\gamma$$





SR: Doppler shift

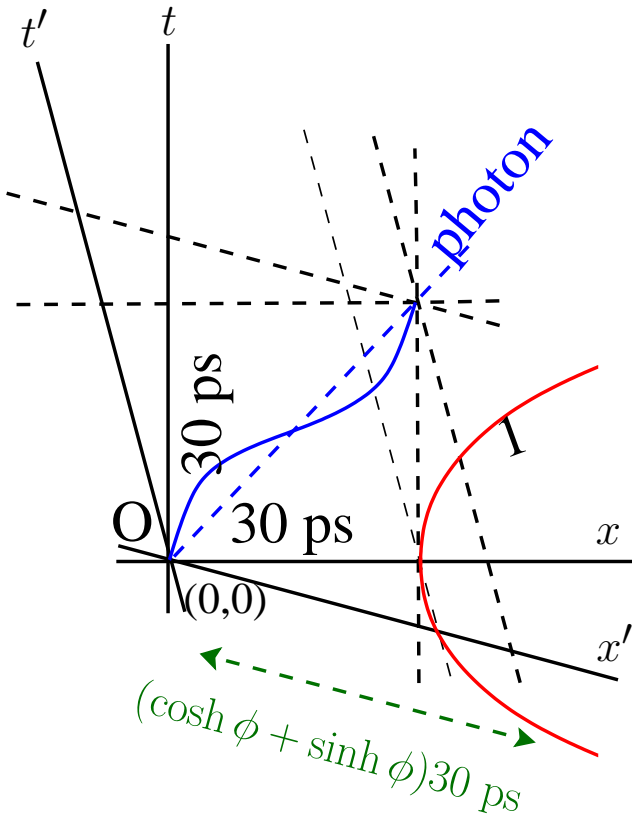


see photon worldline calculation

$$x'/x = \gamma(1 + \beta)$$



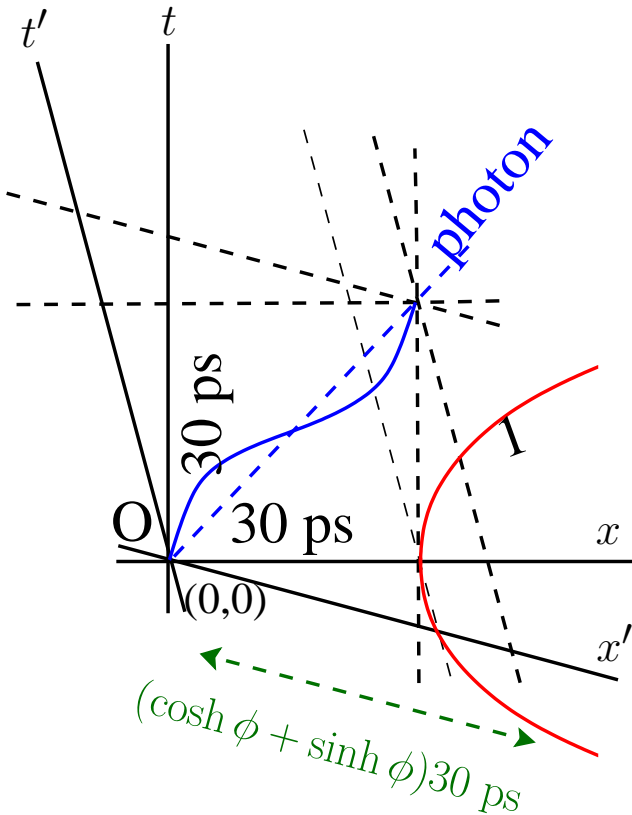
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1 + \beta) = \frac{1 + \beta}{\sqrt{1 - \beta^2}}$$

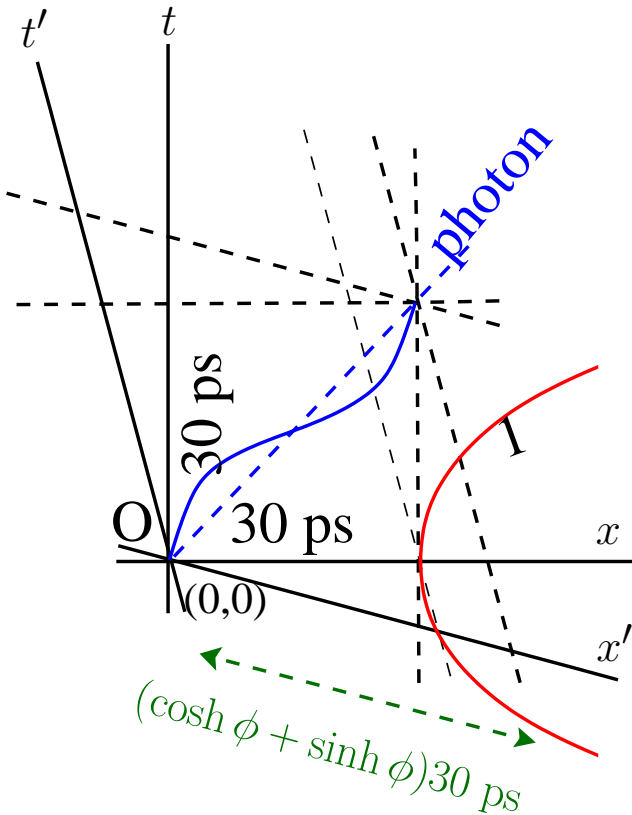
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1 + \beta) = \frac{\sqrt{(1+\beta)^2}}{\sqrt{(1-\beta)(1+\beta)}}$$

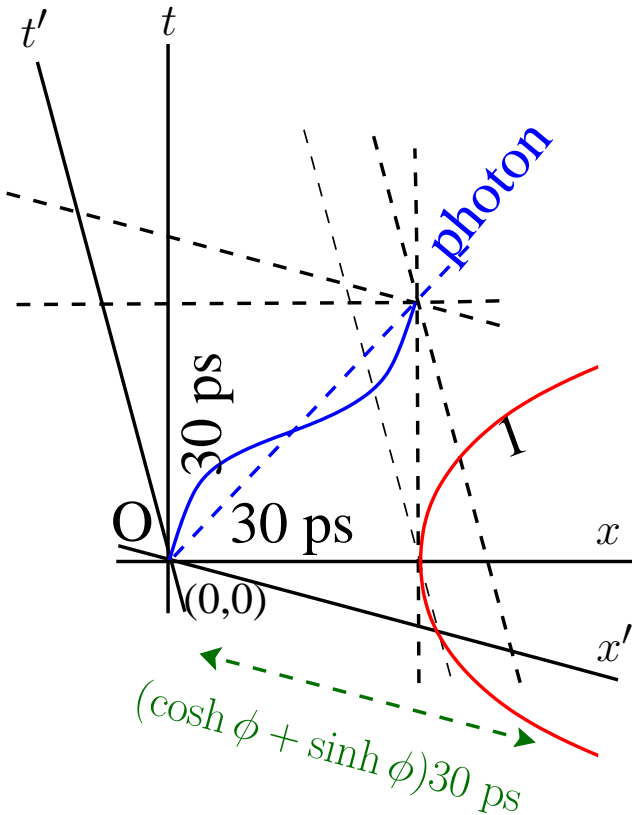
SR: Doppler shift



see photon worldline calculation

$$x'/x = \gamma(1 + \beta) = \sqrt{\frac{1+\beta}{1-\beta}}$$

SR: Doppler shift

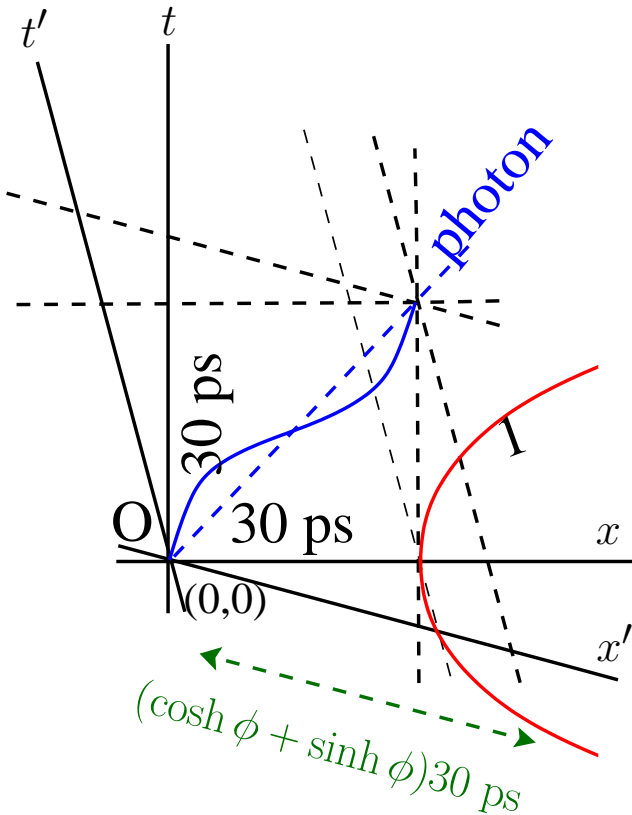


see photon worldline calculation

$$1 + z := \lambda' / \lambda = \gamma(1 + \beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$

redshift

SR: Doppler shift



see photon worldline calculation

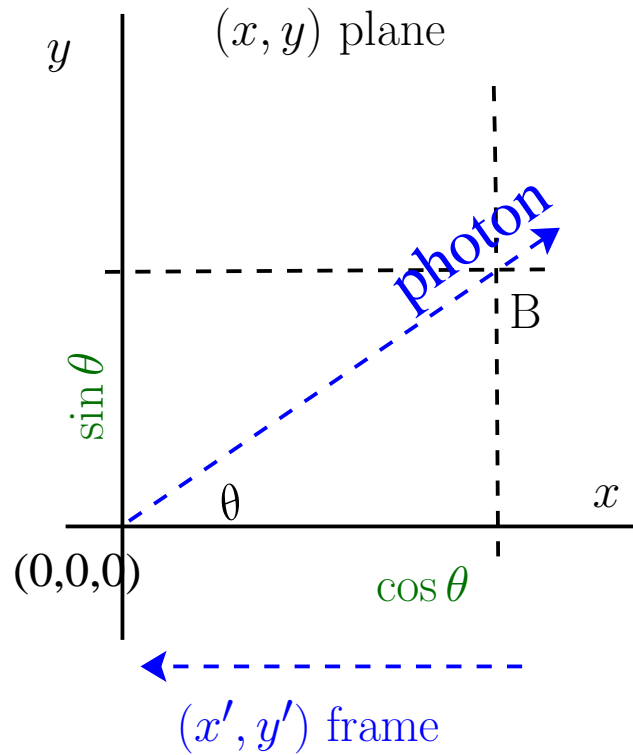
$$1 + z := \lambda' / \lambda = \gamma(1 + \beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}}$$

redshift

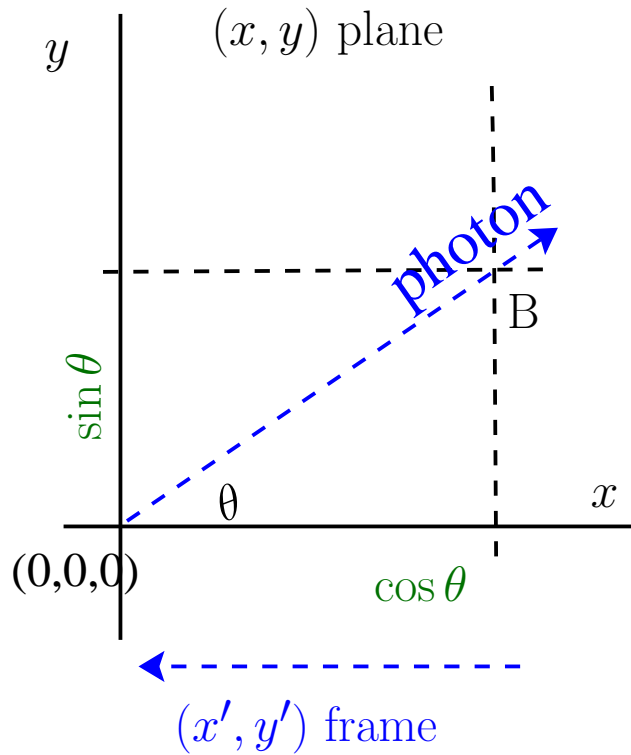
\Rightarrow when $\beta \ll 1$, $z \approx \beta$



SR: relativistic aberration



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event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$



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$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$$



SR: relativistic aberration

event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma & 0 & \beta\gamma \\ 0 & 1 & 0 \\ \beta\gamma & 0 & \gamma \end{pmatrix} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix}$$

SR: relativistic aberration

event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma \cos \theta + \beta \gamma \\ \sin \theta \\ \beta \gamma \cos \theta + \gamma \end{pmatrix}$$

SR: relativistic aberration

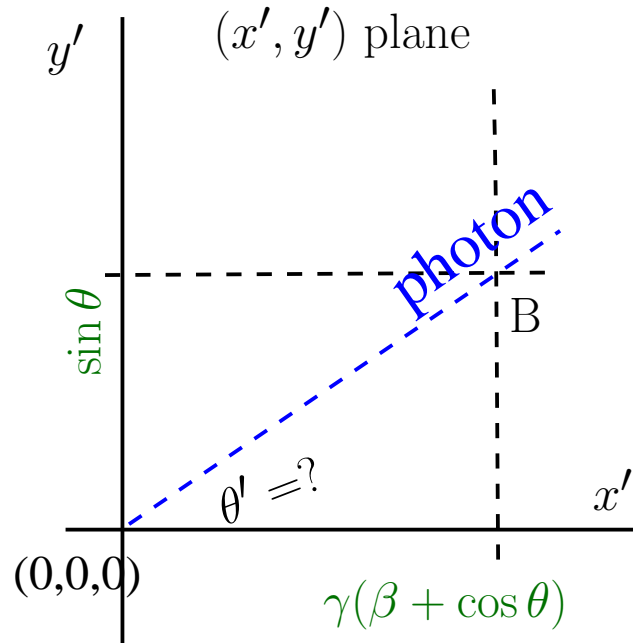
event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$

$$\Lambda^{-1} \begin{pmatrix} \cos \theta \\ \sin \theta \\ 1 \end{pmatrix} = \begin{pmatrix} \gamma(\cos \theta + \beta) \\ \sin \theta \\ \gamma(1 + \beta \cos \theta) \end{pmatrix}$$



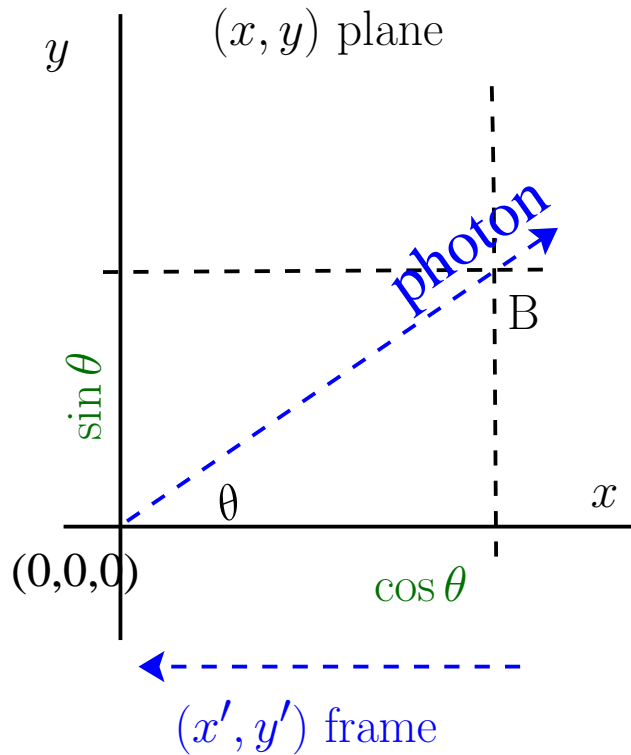
SR: relativistic aberration

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SR: relativistic aberration

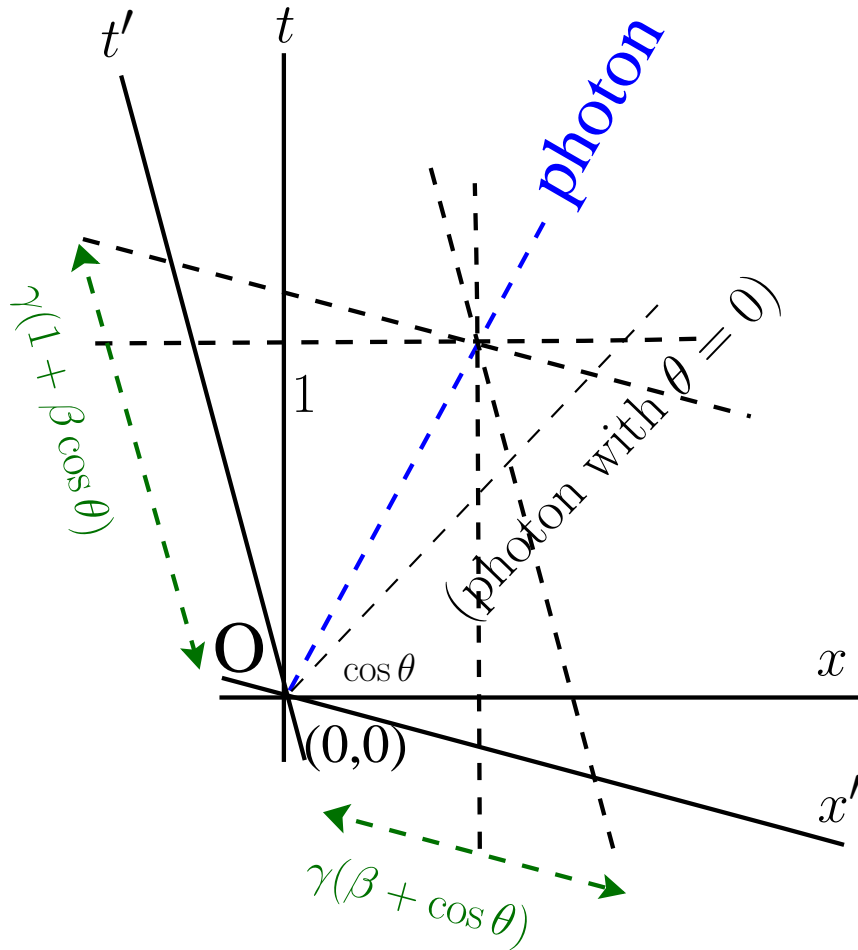
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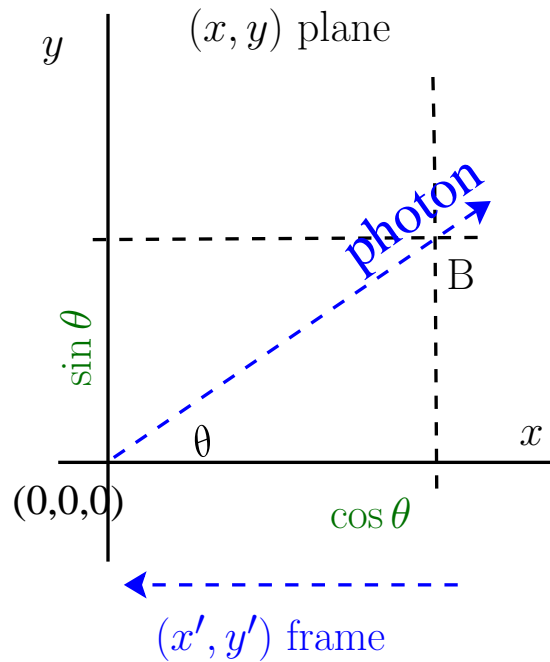
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SR: relativistic aberration



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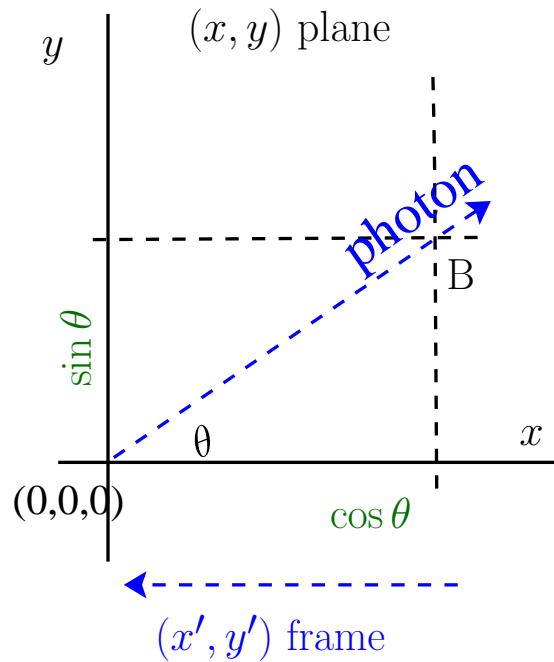
$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)}$$



SR: relativistic aberration



event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$



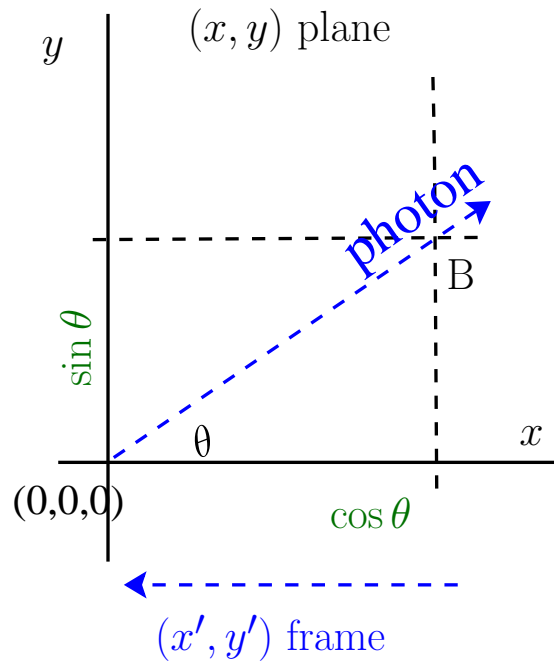
$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)} < \tan \theta \text{ if } 0 < \beta < 1$$

w: Relativistic aberration (2011-02-22: quality=weak)



SR: relativistic aberration

event B: $(x, y, t) = (\cos \theta, \sin \theta, 1)$



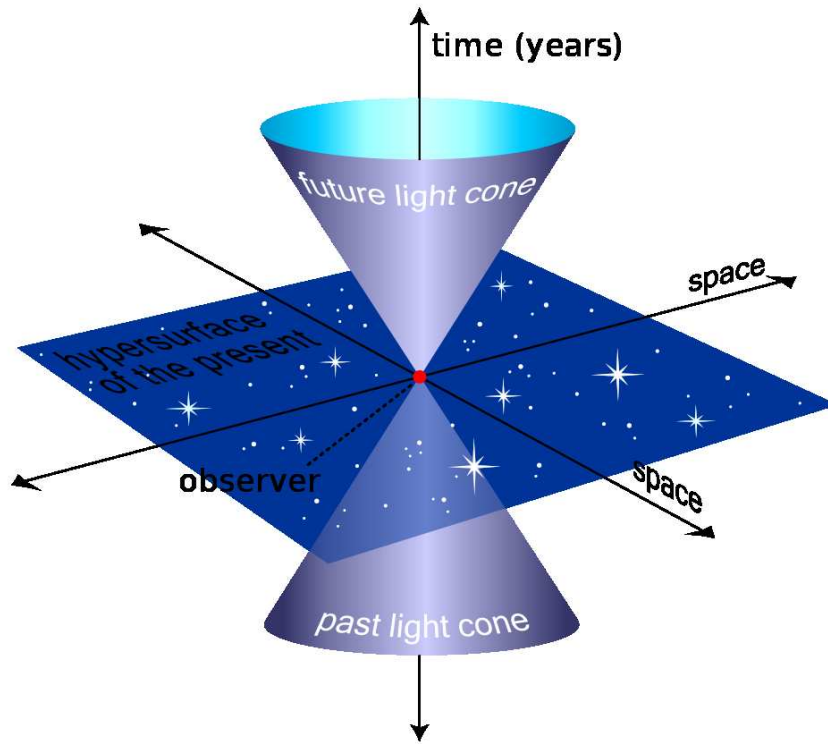
$$\tan \theta' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)} < \tan \theta \text{ if } 0 < \beta < 1$$

w: Relativistic aberration (2011-02-22: quality=weak)

⇒ relativistic beaming, e.g. AGN jets

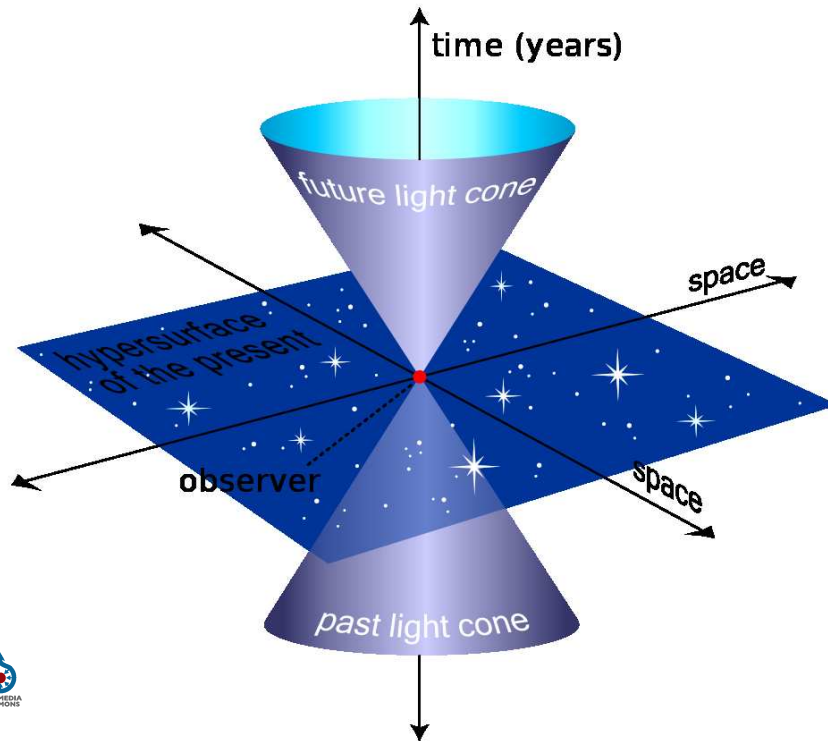


SR: world line





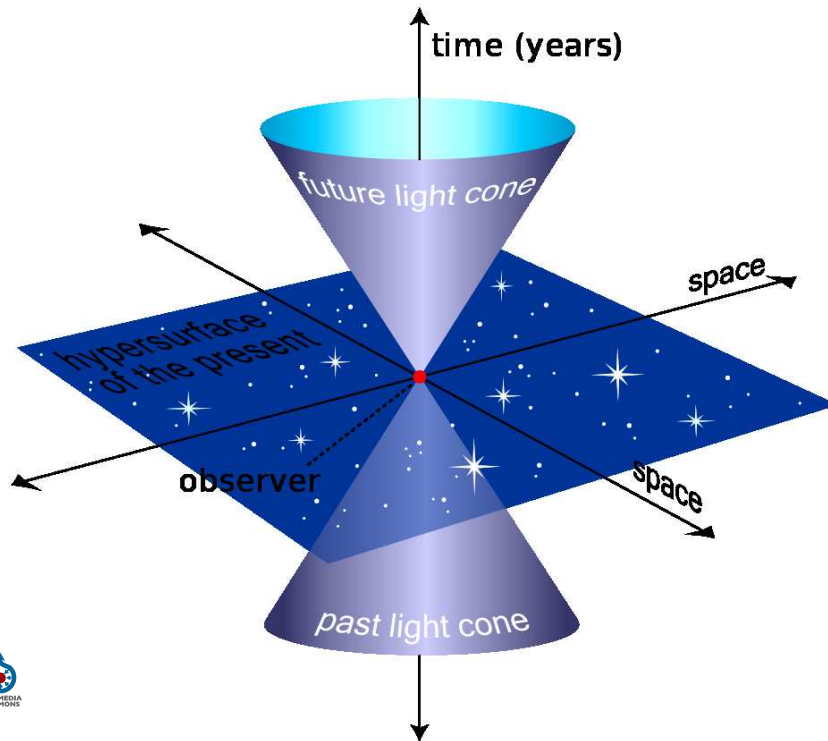
SR: world line



lightlike interval = null interval: $(\Delta s)^2 = 0$
 spacetime =



SR: world line

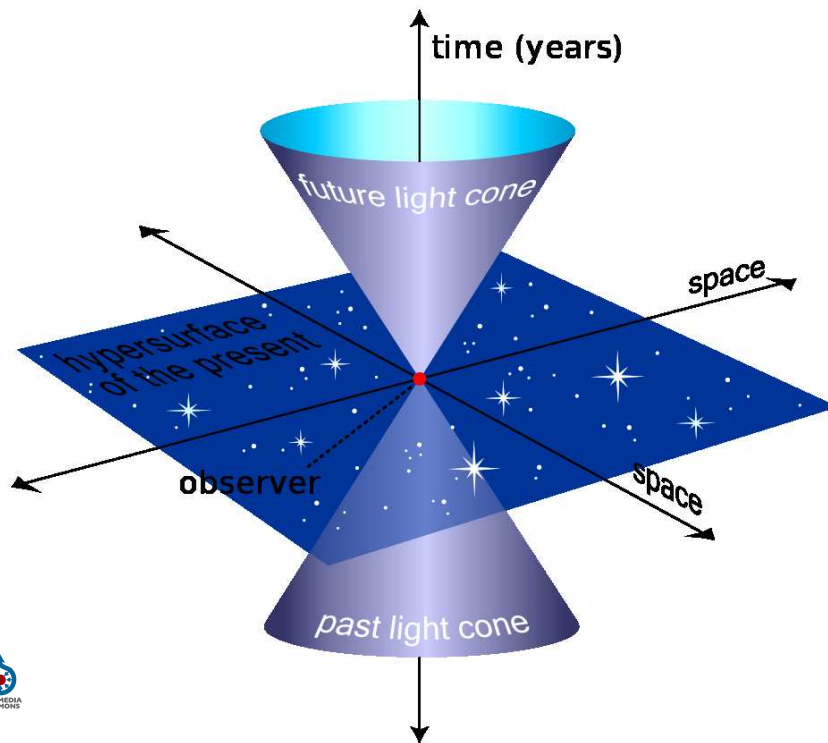


lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone



SR: world line

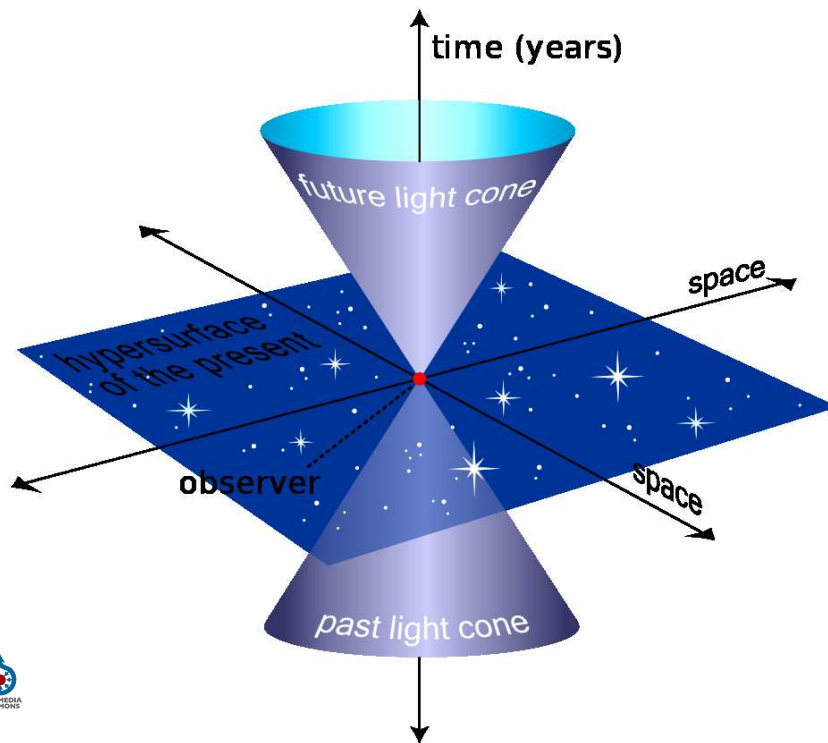


lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone
 + on future light cone + inside future light cone



SR: world line



lightlike interval = null interval: $(\Delta s)^2 = 0$

spacetime = on past w:light cone + inside past light cone

+ on future light cone + inside future light cone

+ elsewhere



SR: world line

Lorentz transform of world line





SR: world line

Lorentz transform of world line



SR: world line



Lorentz transform of world line



- coordinate time in spacetime model \neq time in your brain (thinking)



SR: world line



Lorentz transform of world line



- coordinate time in spacetime model \neq time in your brain (thinking)
- $\frac{dt}{dt_{\text{thinking}}}$ can be positive or negative



SR: world line



Lorentz transform of world line



- coordinate time in spacetime model \neq time in your brain (thinking)
- $\frac{dt}{d\lambda}$ can be positive or negative, λ arbitrary real parameter



SR: world line



Lorentz transform of world line



- coordinate time in spacetime model \neq time in your brain (thinking)
- $\frac{dt}{d\lambda}$ can be positive or negative, λ arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though $\frac{dt}{d\lambda} > 0$



SR: world line

Lorentz transform of world line



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- w:proper time $\tau :=$ time along a worldline measured by clock following that worldline

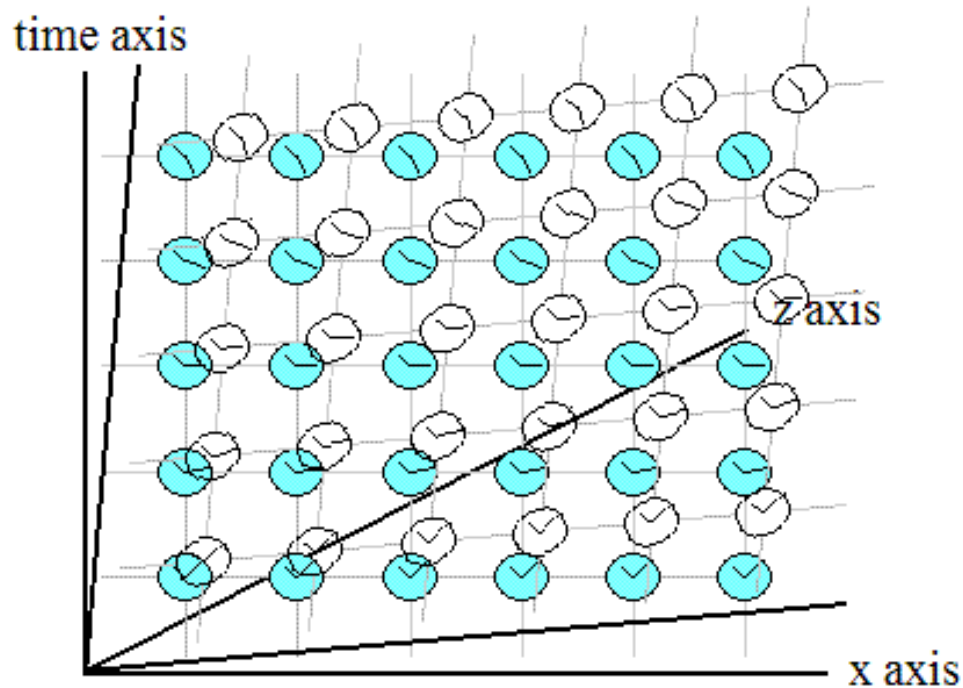
SR: world line

Lorentz transform of world line



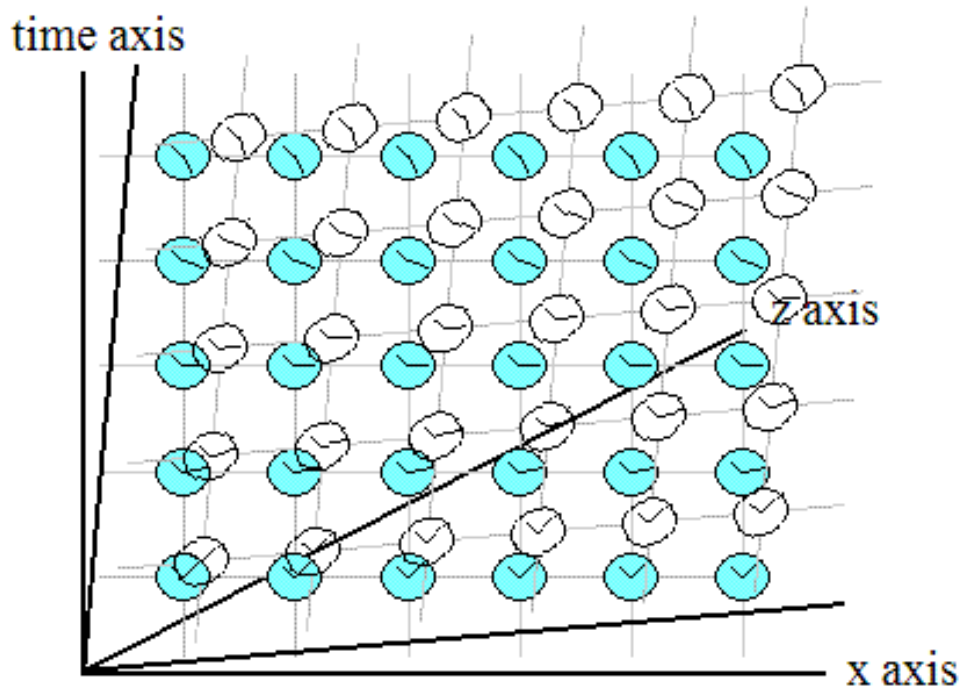
- coordinate time in spacetime model \neq time in your brain (thinking)
- $\frac{dt}{d\lambda}$ can be positive or negative, λ arbitrary real parameter
- “elsewhere” spacetime events can change from past to future even though $\frac{dt}{d\lambda} > 0$
- w:proper time $\tau :=$ time along a worldline measured by clock following that worldline
- often $d\tau$ is useful for integrating

SR: Rietdijk–Putnam–Penrose p.



Relativity shows that the inertial frames of reference of relatively moving objects do not overlie each other.

SR: Rietdijk–Putnam–Penrose p.



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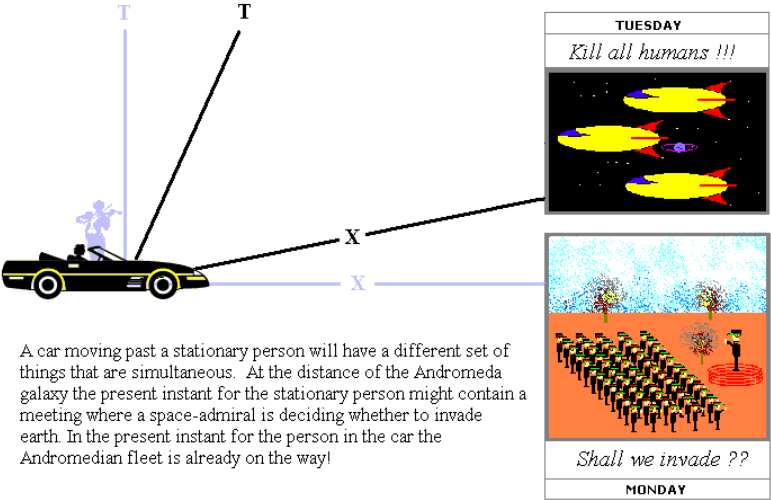
b:Inertialoverlay.GIF

- each observer can synchronise clocks + rods



SR: Rietdijk–Putnam–Penrose p.

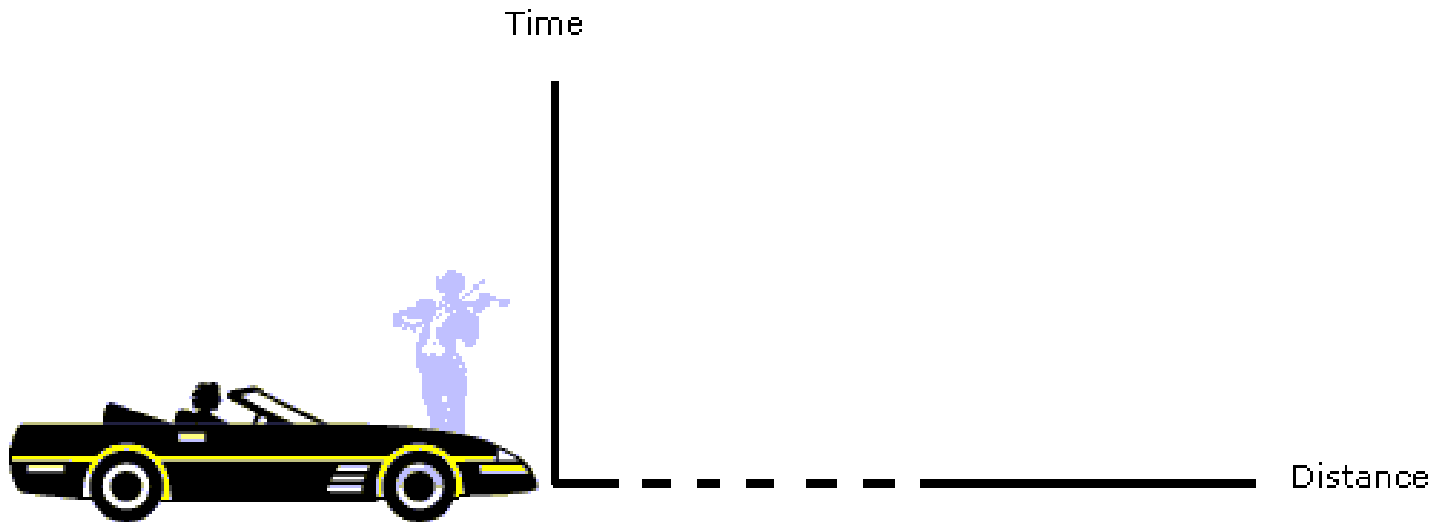
The Andromeda Paradox



w:Rietdijk-Putnam argument b:Rel2.gif



SR: Rietdijk–Putnam–Penrose p.



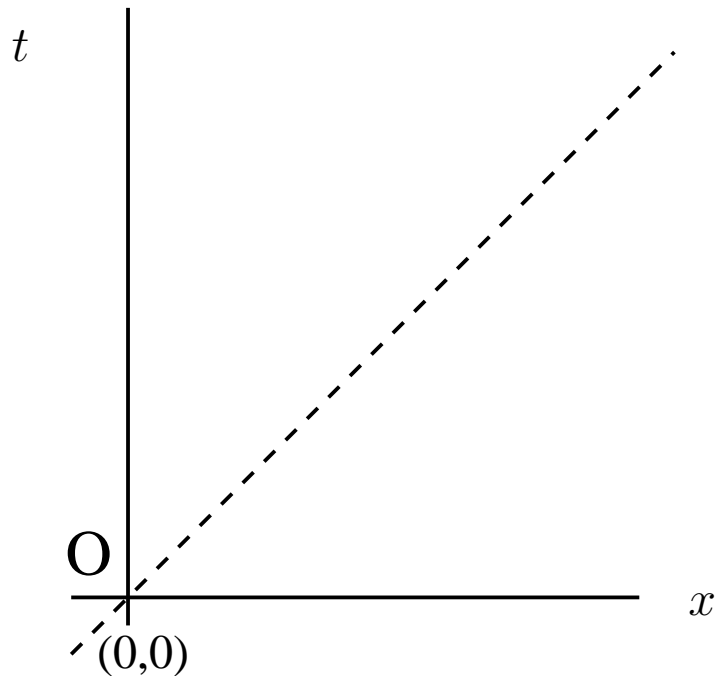
For the car driver the stationary man and the invasion fleet are all events in the present moment.

[w:Rietdijk-Putnam argument](#) [b:Rel3.gif](#)





SR: tachyons and causality

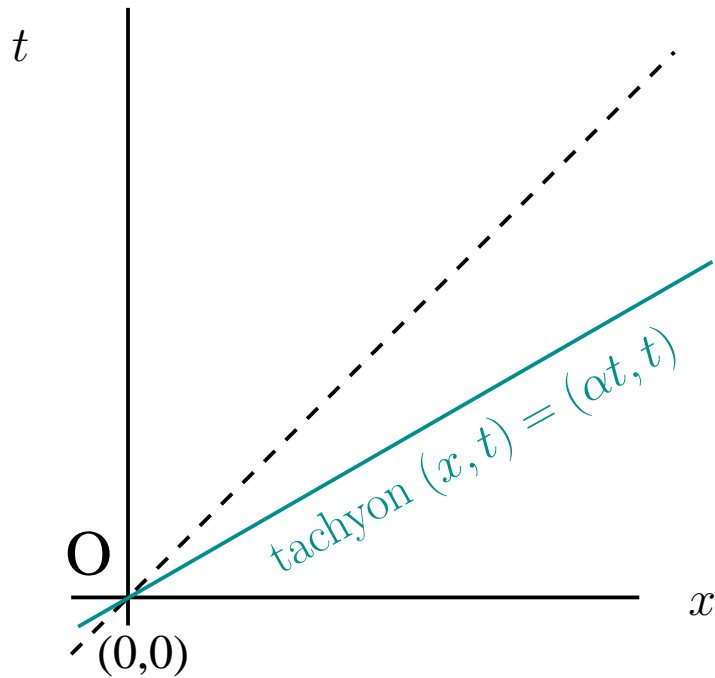


observer "at rest"





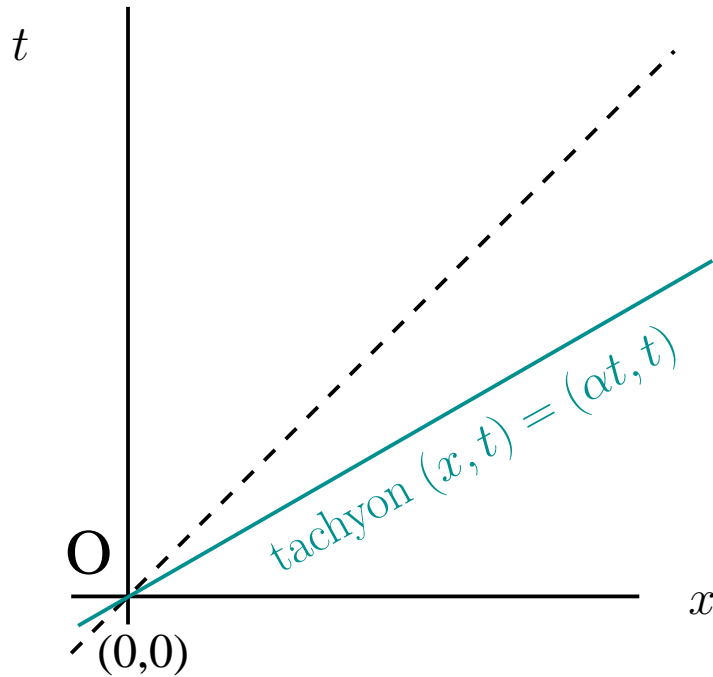
SR: tachyons and causality



add a tachyon with speed $\alpha > 1$



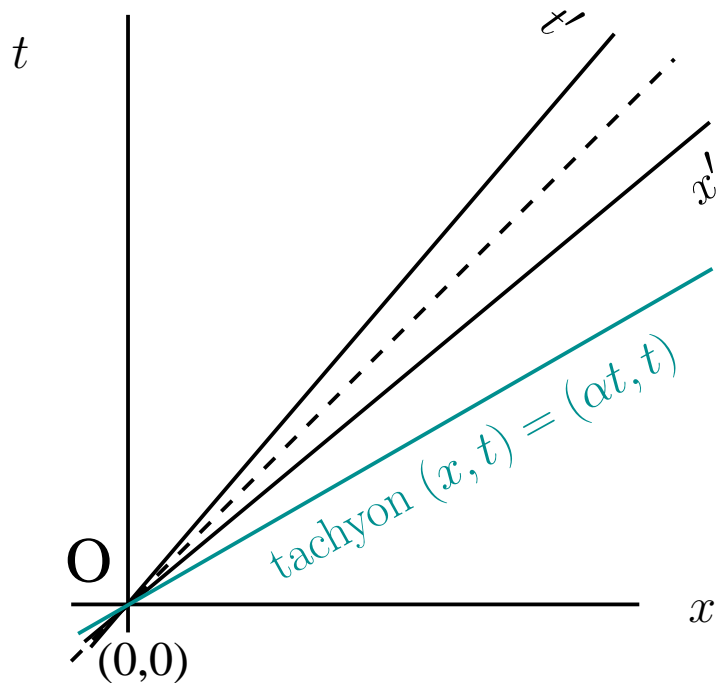
SR: tachyons and causality



add a tachyon with speed $\alpha > 1$

choose rocket at speed β with $1/\alpha < \beta < 1$

SR: tachyons and causality

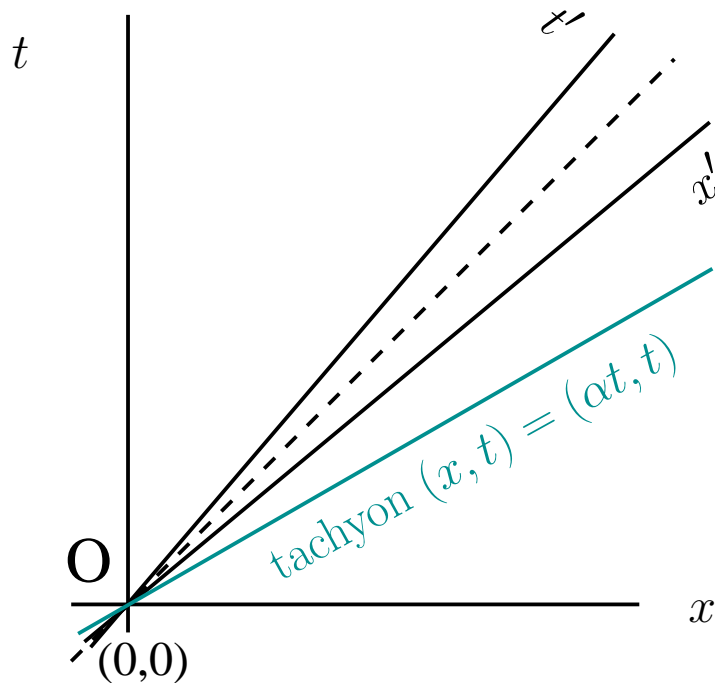


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add axes x', t' for the rocket

SR: tachyons and causality



add a tachyon with speed $\alpha > 1$

choose rocket at speed β with $1/\alpha < \beta < 1$

add axes x', t' for the rocket

rocket frame: $(\alpha t, t)$ becomes $\Lambda (\alpha t, t)^T$

SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} =$$



SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \begin{pmatrix} \gamma \alpha t - \beta \gamma t \\ -\alpha \beta \gamma t + \gamma t \end{pmatrix}$$



SR: tachyons and causality

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t \\ t \end{pmatrix} = \gamma t \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$



SR: tachyons and causality

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$$x' = \gamma t(\alpha - \beta) > 0 \text{ since } \alpha > 1 > \beta$$



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$$dt'/dt < 0$$

same sequence of spacetime events = tachyon worldline:

t increases for observer “at rest”,

t' decreases for rocket observer (with $\beta > 1/\alpha$)

SR: tachyons and causality

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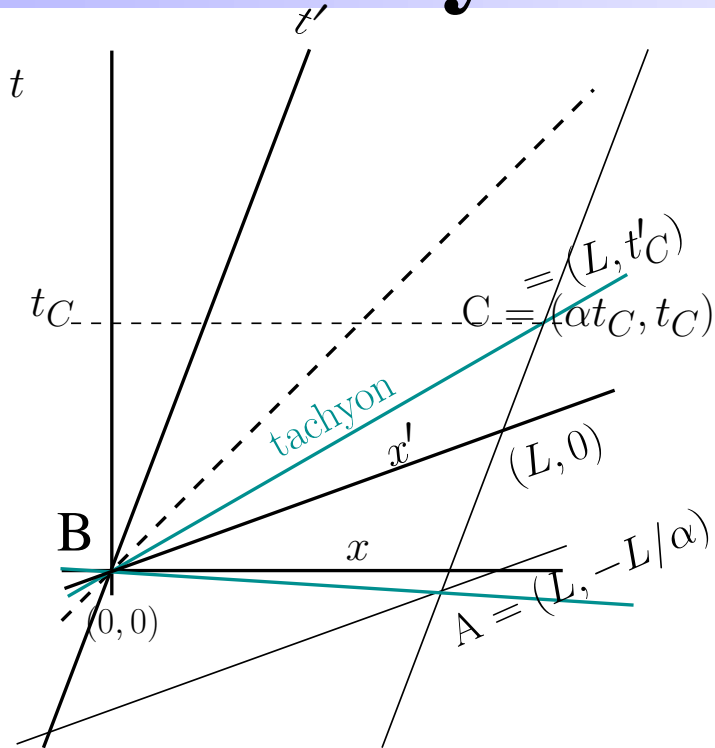
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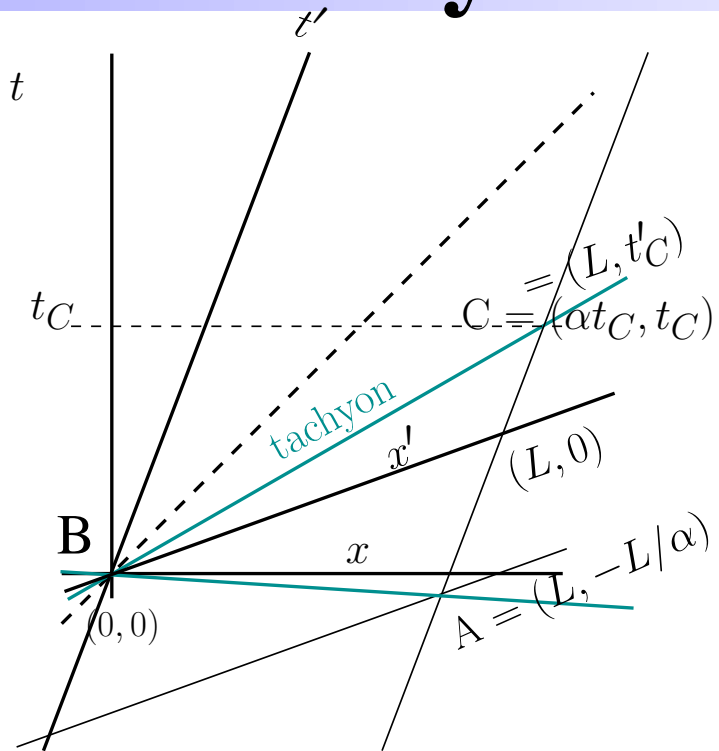
t' decreases for rocket observer (with $\beta > 1/\alpha$)

- observer at rest: tachyon emitted at origin
- rocket: tachyon absorbed at origin

SR: tachyonic antitelephone

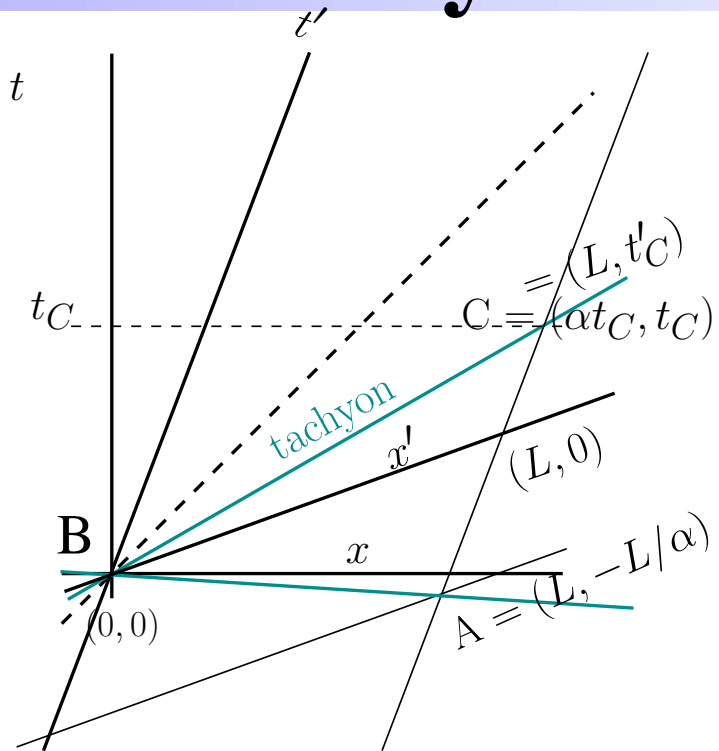


SR: tachyonic antitelephone



B stationary: (x, t)
frame

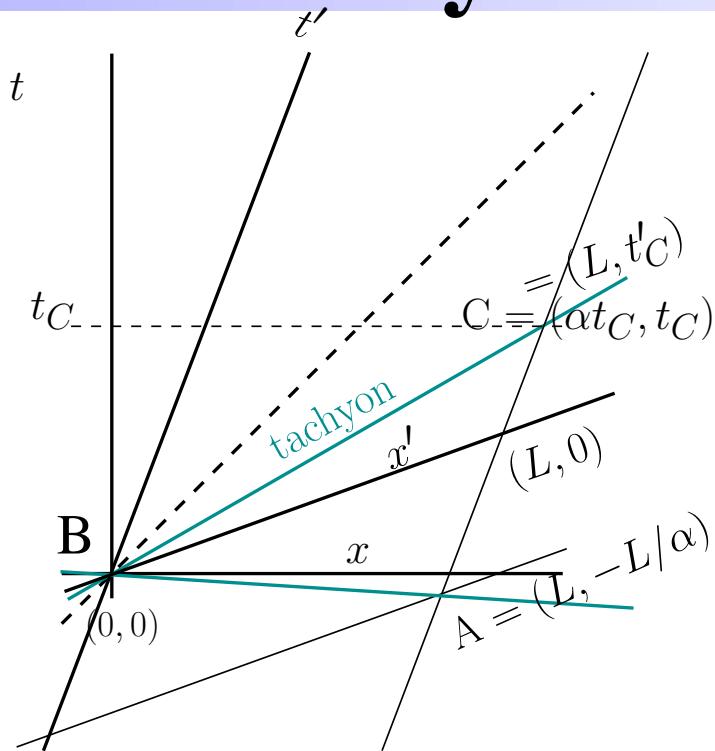
SR: tachyonic antitelephone



B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone

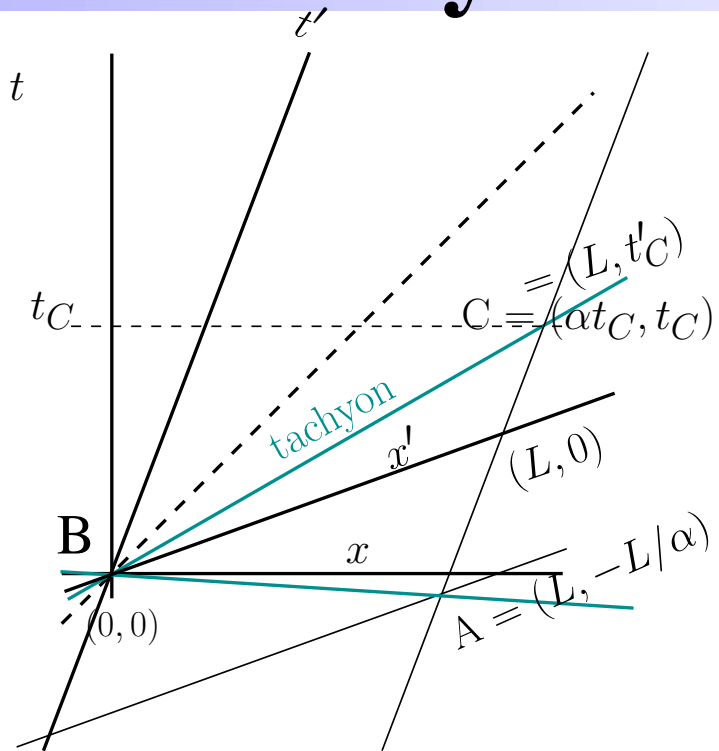


A: tachyon at $\alpha > 1$ to B

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone

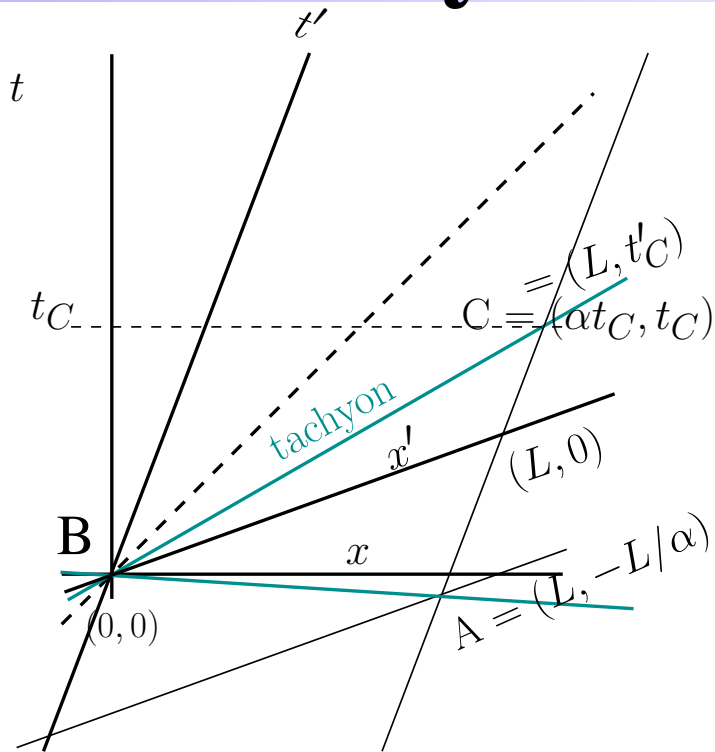


B: tachyon at $\alpha > 1$ to C

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone

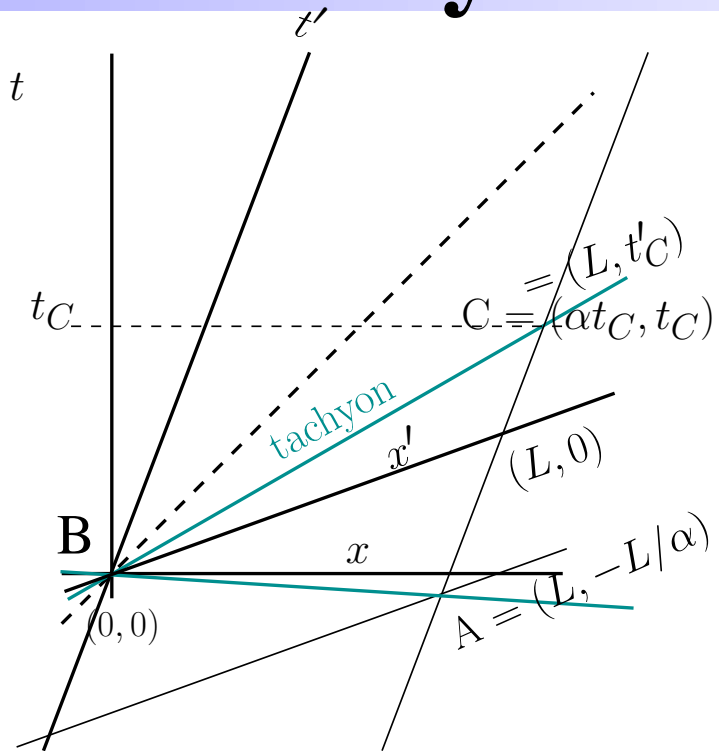


$$C: \begin{pmatrix} L \\ t'_C \end{pmatrix} = \Lambda \begin{pmatrix} \alpha t_C \\ t_C \end{pmatrix}$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone

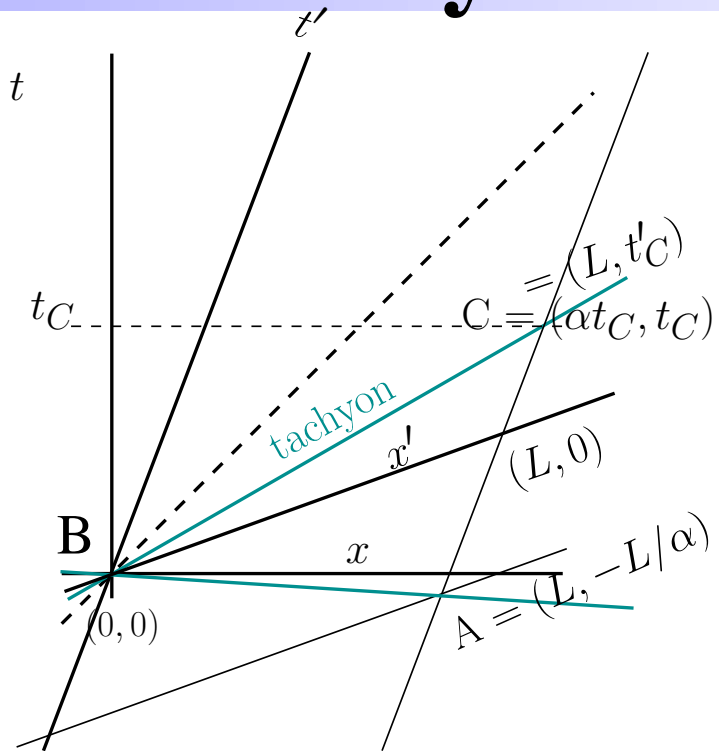


$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone



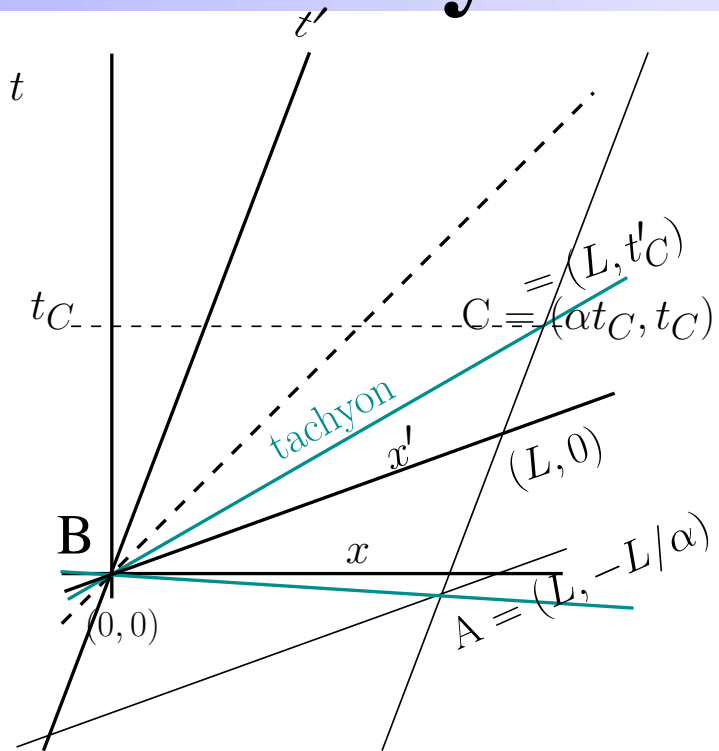
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma t_C (1 - \alpha\beta)$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone



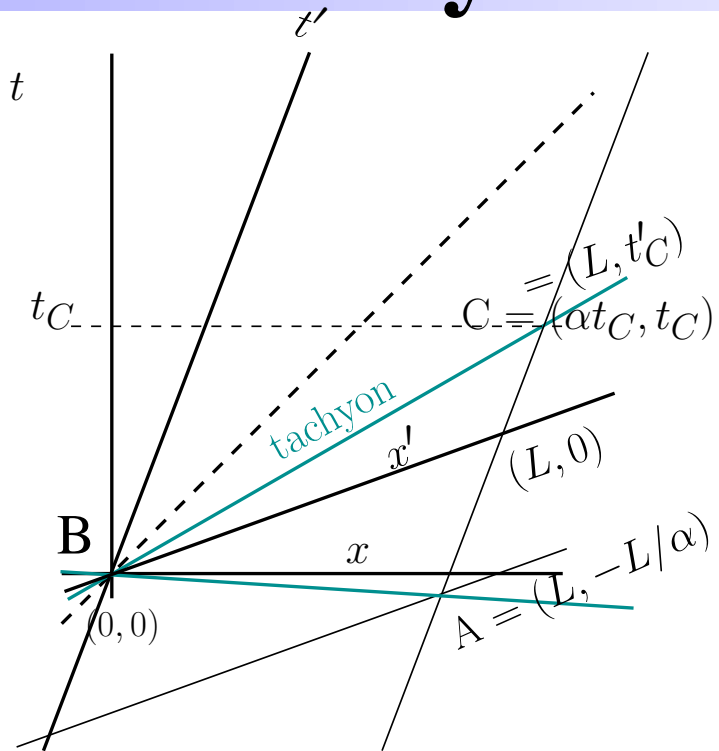
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = \gamma \frac{L}{\gamma(\alpha - \beta)} (1 - \alpha\beta)$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone



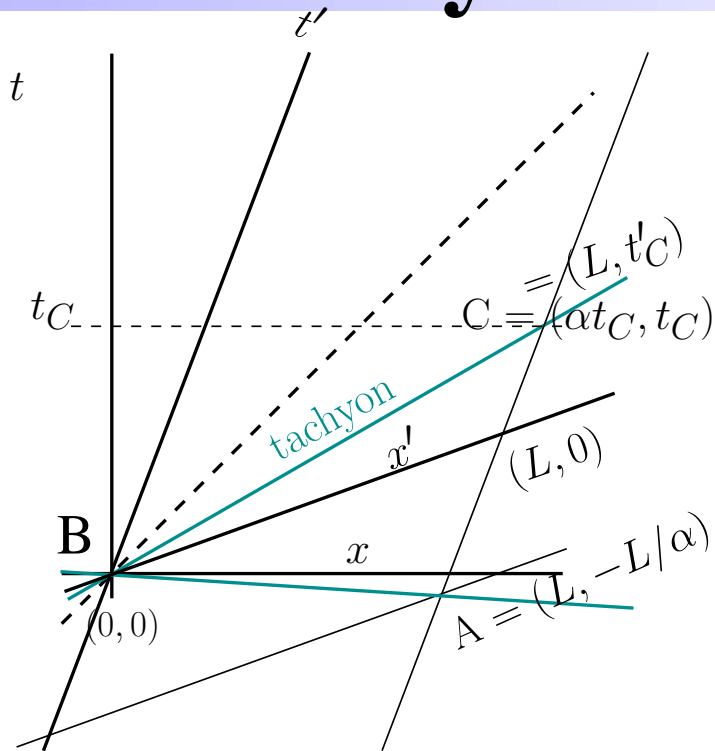
$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone



$$\begin{pmatrix} L \\ t'_C \end{pmatrix} = \gamma t_C \begin{pmatrix} \alpha - \beta \\ -\alpha\beta + 1 \end{pmatrix}$$

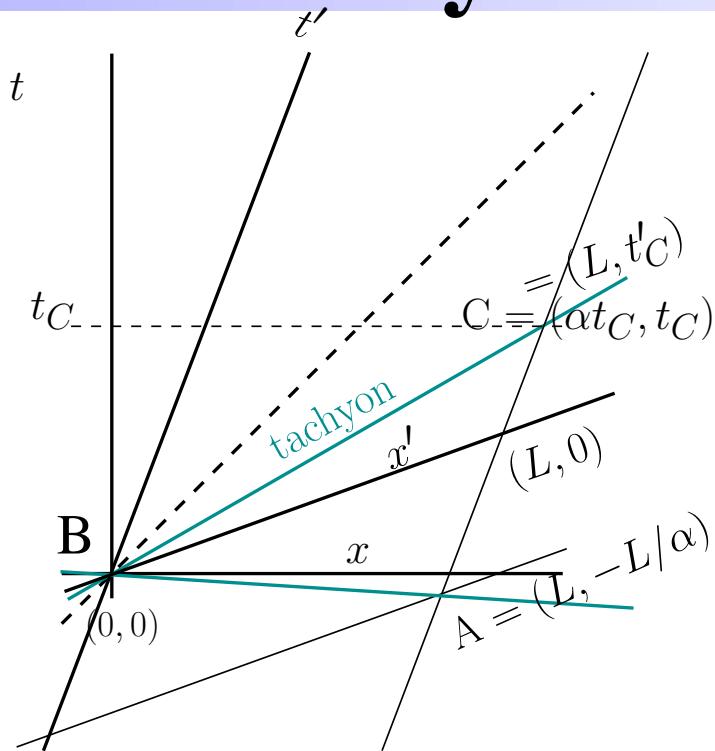
$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

$$t'_C - t'_A = L \left(\frac{1 - \alpha\beta}{\alpha - \beta} + \frac{1}{\alpha} \right)$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

SR: tachyonic antitelephone



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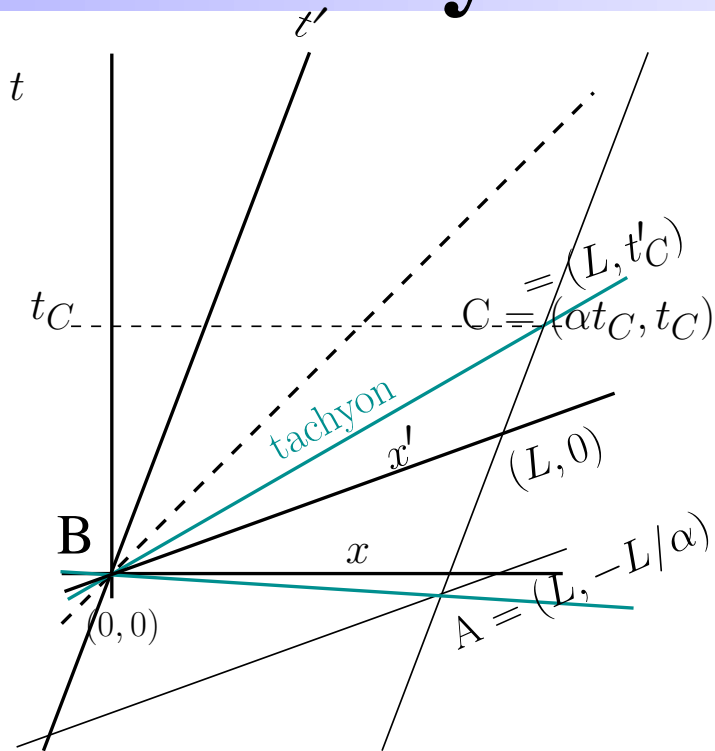
$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

$$t'_C - t'_A = L \frac{\alpha - \alpha^2\beta + \alpha - \beta}{\alpha(\alpha - \beta)}$$

B stationary: (x, t)
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A moving at speed β :
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SR: tachyonic antitelephone



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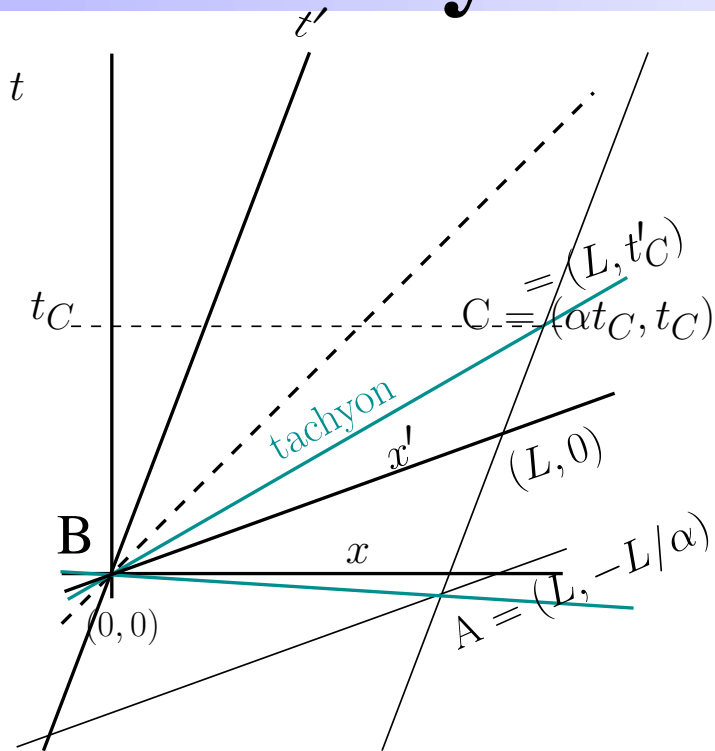
$$t'_C = L \frac{1 - \alpha\beta}{\alpha - \beta}$$

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A moving at speed β :
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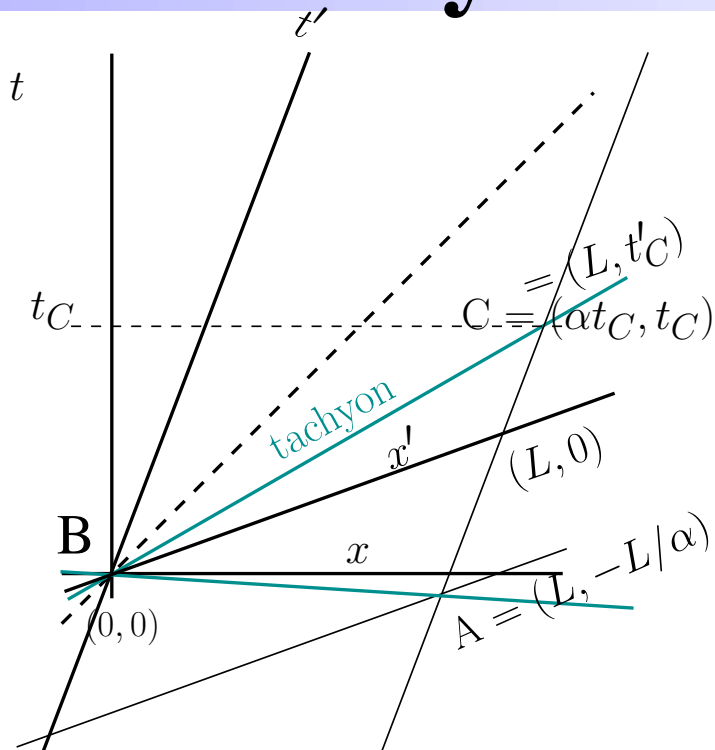
$$t'_C - t'_A = L \frac{2\alpha - (\alpha^2 + 1)\beta}{\alpha(\alpha - \beta)}$$

$$< 0 \text{ if } \beta > \frac{2\alpha}{\alpha^2 + 1}$$

B stationary: (x, t)
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A moving at speed β :
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SR: tachyonic antitelephone



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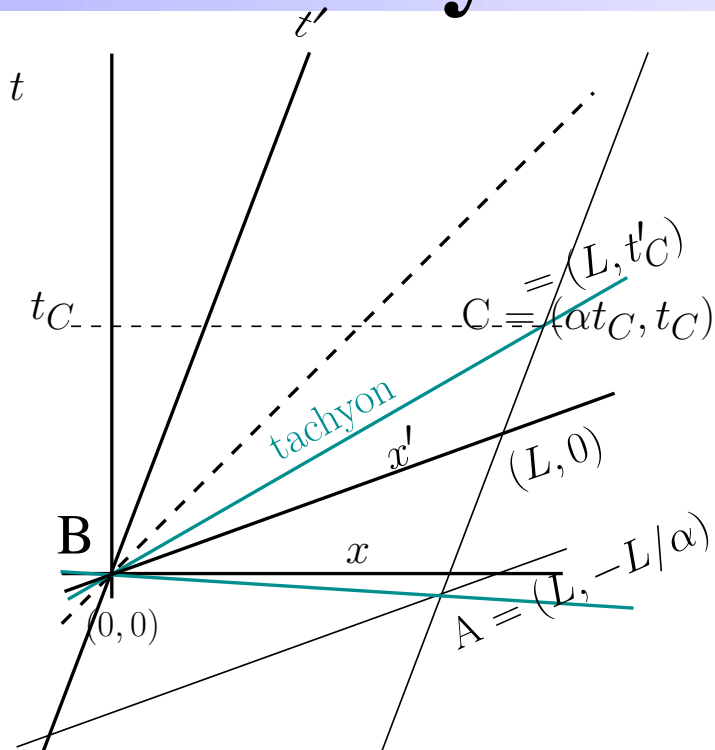
$$< 0 \text{ if } \beta > \frac{2\alpha}{\alpha^2 + 1}$$

B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

A receives tachyonic response at C before sending it

SR: tachyonic antitelephone



B stationary: (x, t)
frame

A moving at speed β :
 (x', t') frame

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$$t'_C - t'_A = L \frac{2\alpha - (\alpha^2 + 1)\beta}{\alpha(\alpha - \beta)}$$

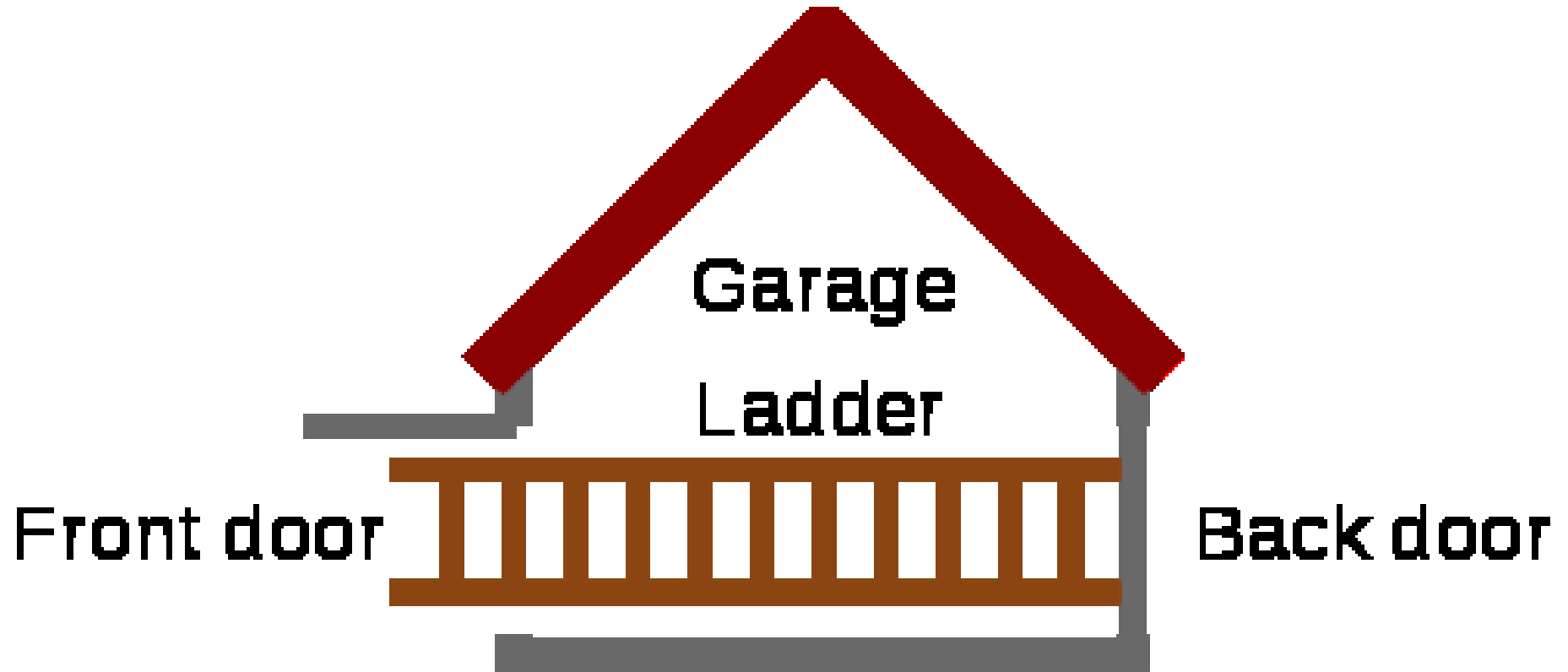
$$< 0 \text{ if } \beta > \frac{2\alpha}{\alpha^2 + 1}$$

A receives tachyonic response at C before sending it

w:tachyonic antitelephone



SR: pole-barn/ladder paradox



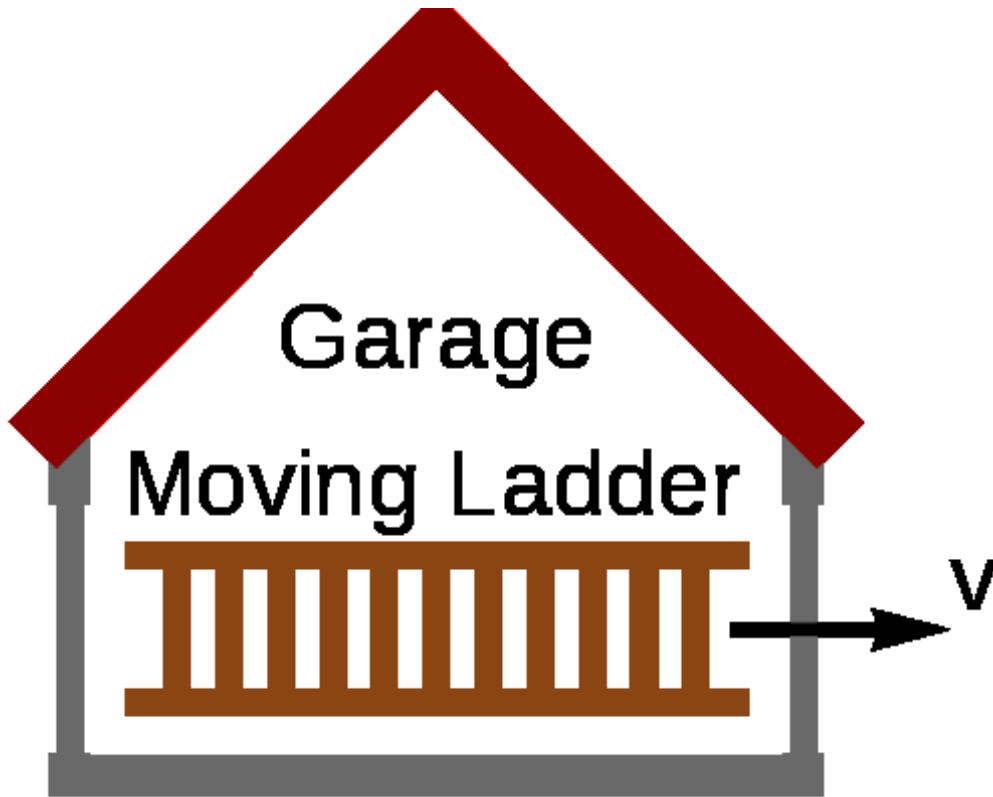
SR: pole-barn/ladder paradox



- ladder of length 29.9γ ns, garage length 30 ns

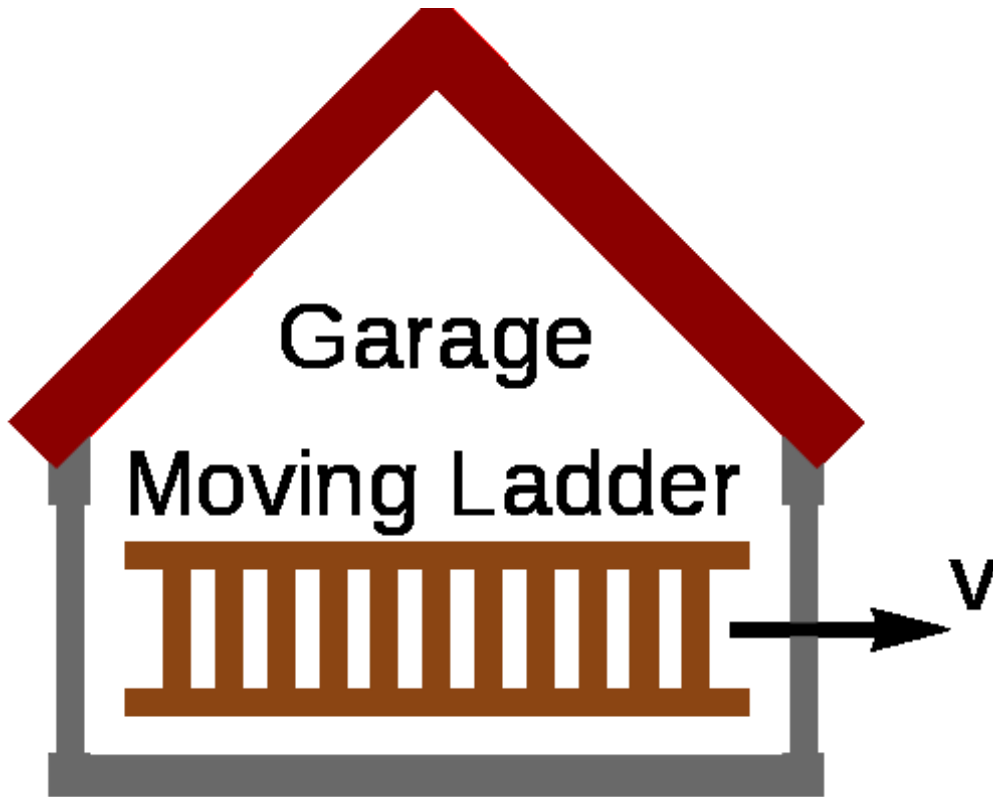


SR: pole-barn/ladder paradox



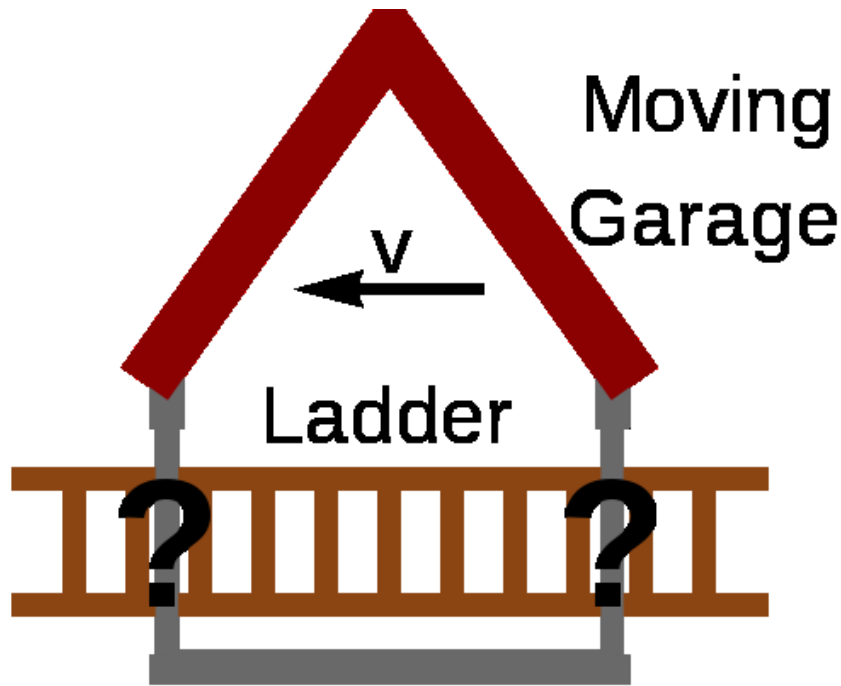
- ladder of length 29.9γ ns, garage length 30 ns
- instantaneously close front + back doors

SR: pole-barn/ladder paradox



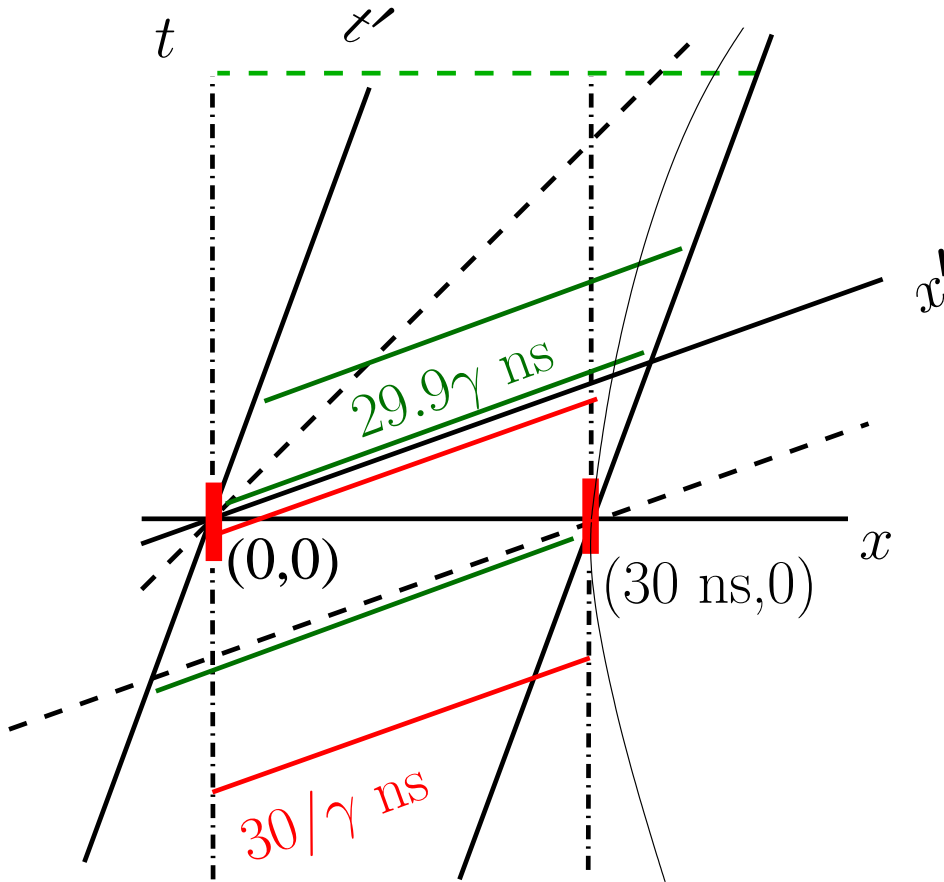
- ladder of length 29.9γ ns, garage length 30 ns
- instantaneously close front + back doors
- 29.9γ ns / $\gamma < 30$ ns \Rightarrow OK

SR: pole-barn/ladder paradox



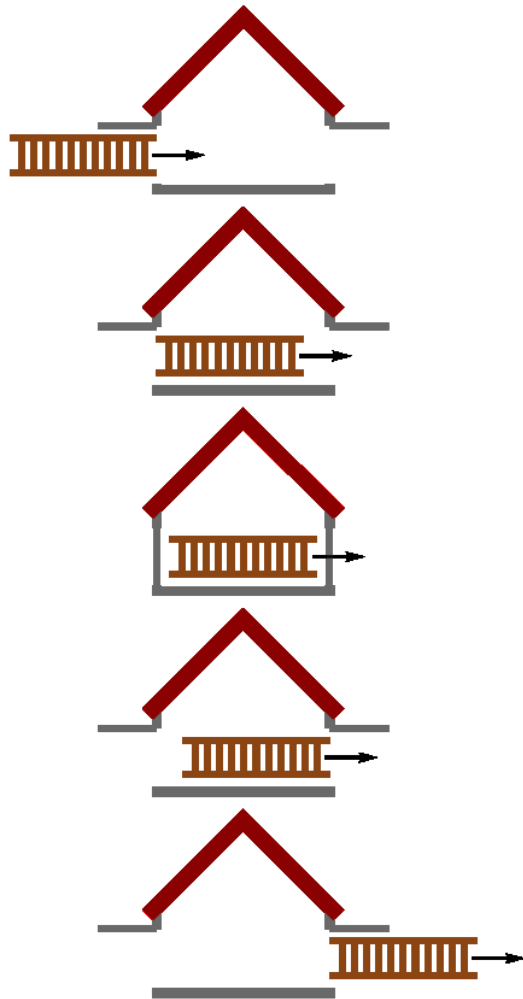
- ladder of length 29.9γ ns, garage length 30 ns
 - instantaneously close front + back doors
 - ladder frame: garage $30/\gamma$ ns long $\ll 29.9\gamma$ ns!!
- Is this possible or not? Make a spacetime diagram.

SR: pole-barn/ladder paradox



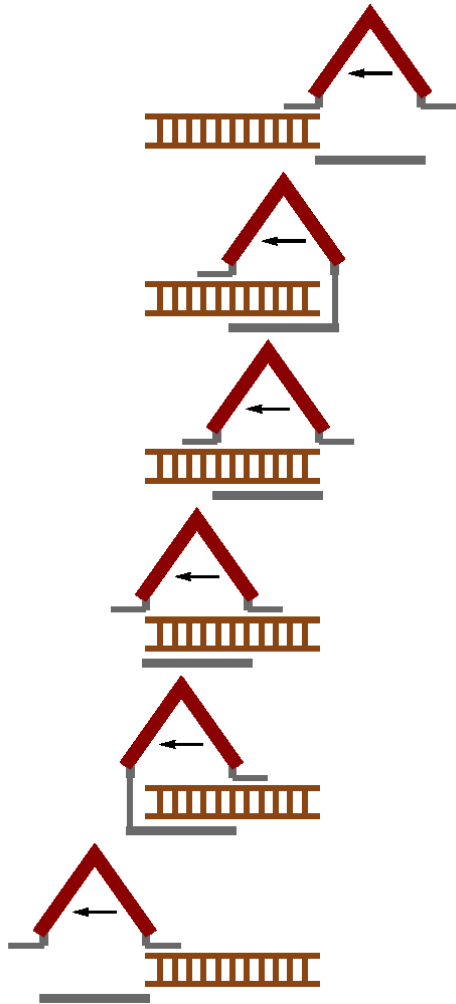


SR: pole-barn/ladder paradox



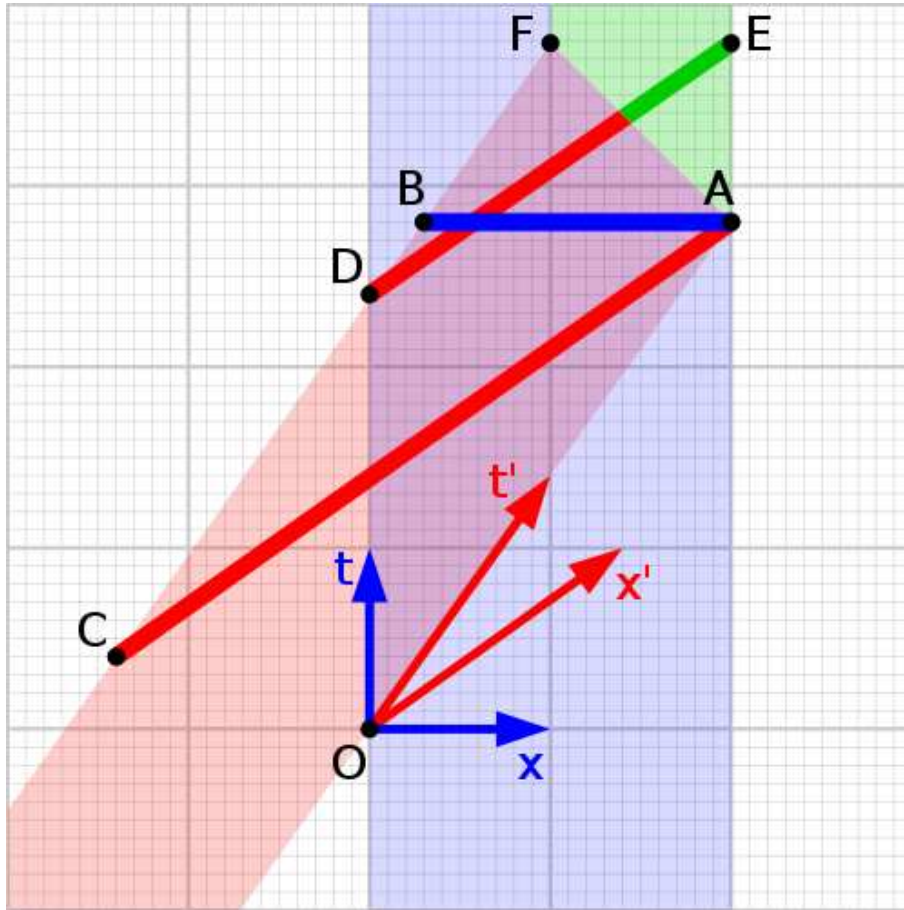


SR: pole-barn/ladder paradox





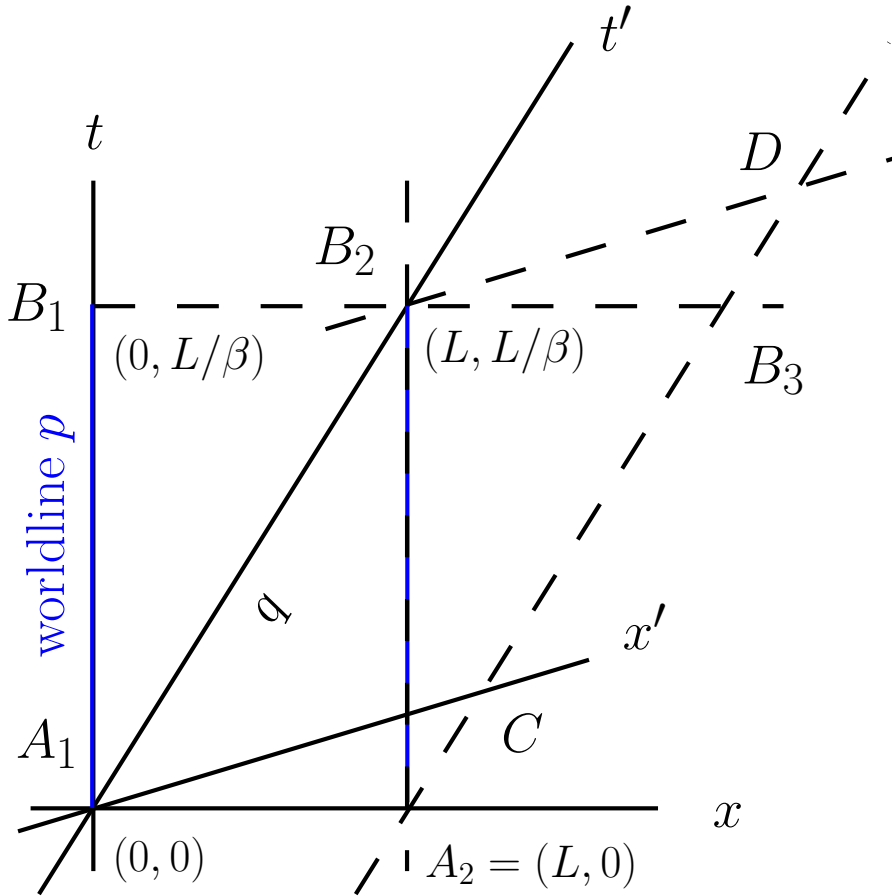
SR: pole-barn/ladder paradox



w:Ladder paradox

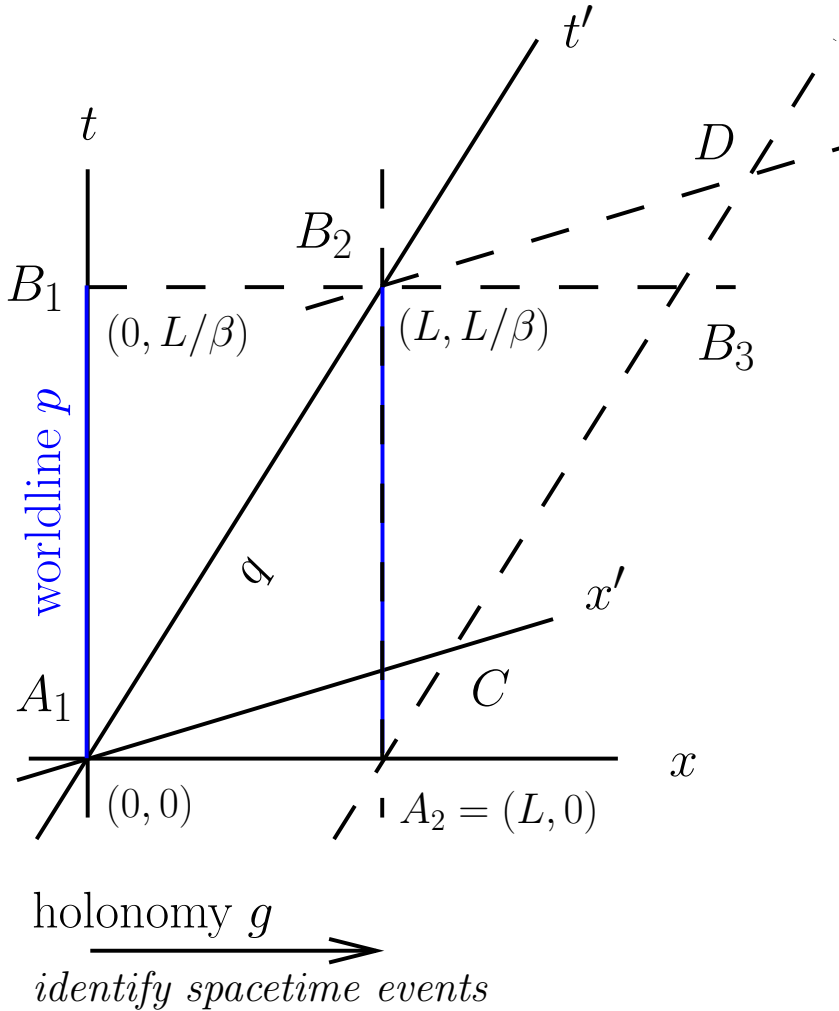


SR: twins paradox



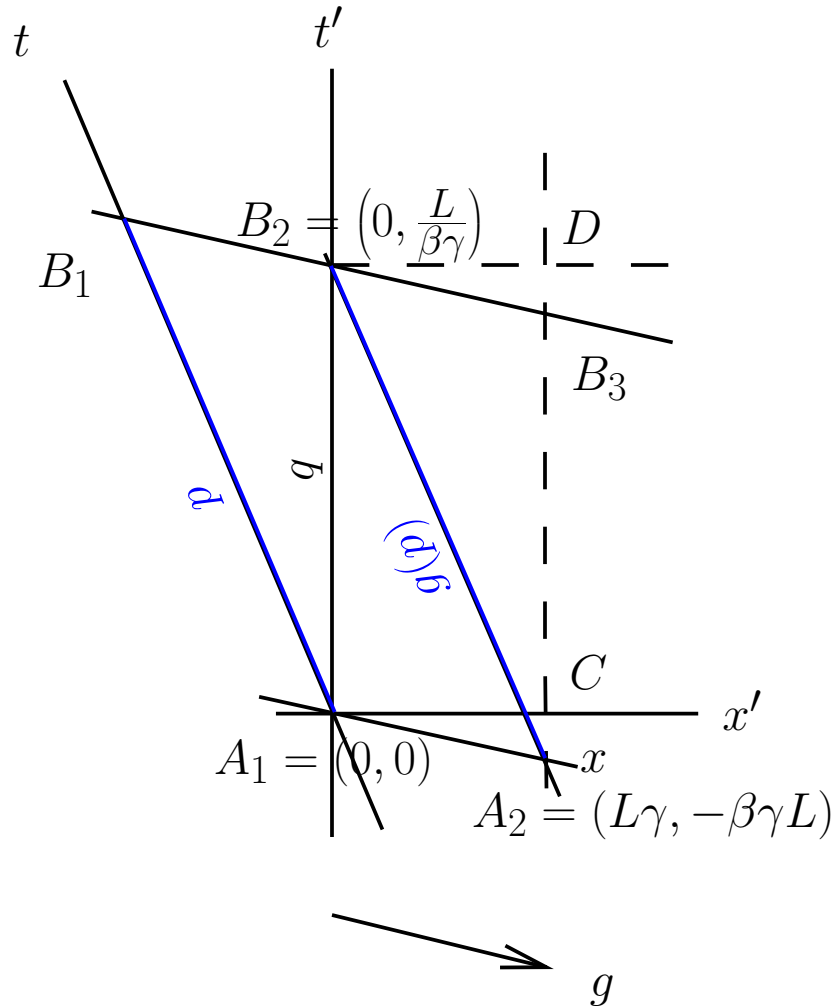
simply connected Minkowski

SR: twins paradox



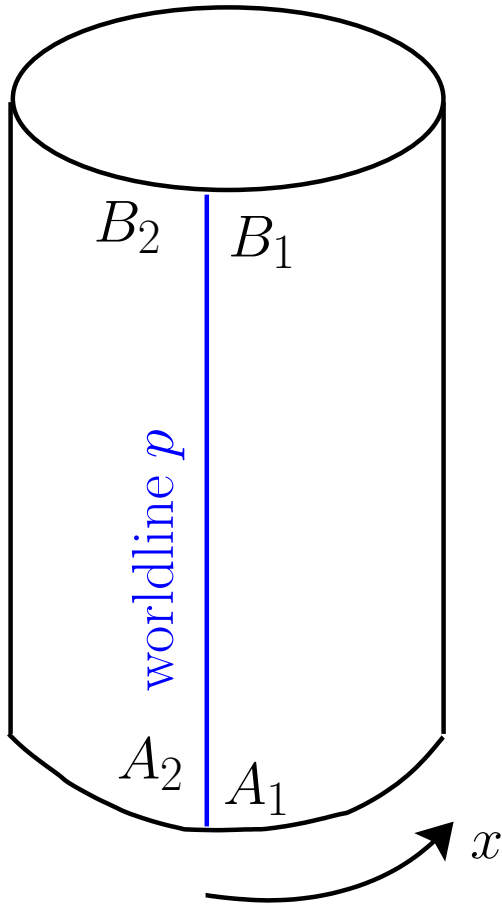


SR: twins paradox



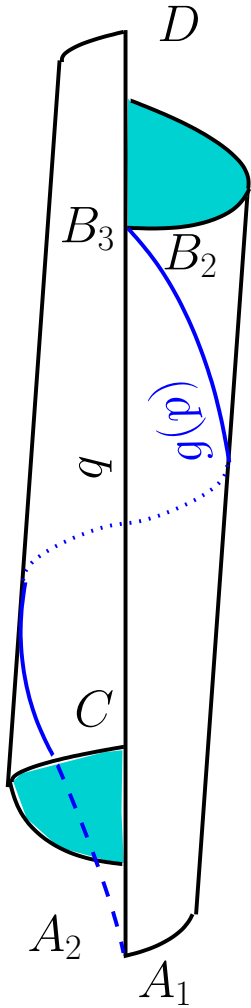


SR: twins paradox



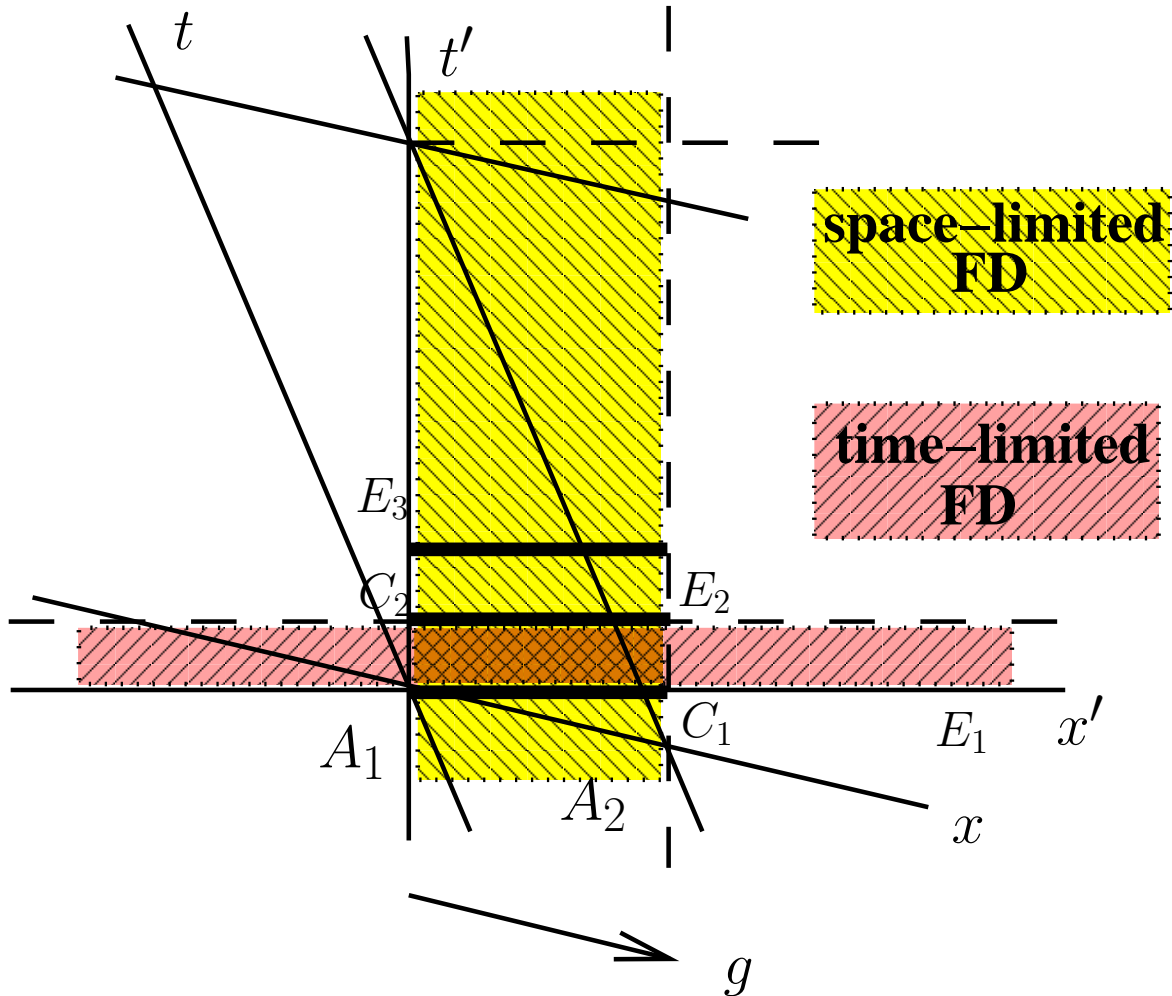


SR: twins paradox

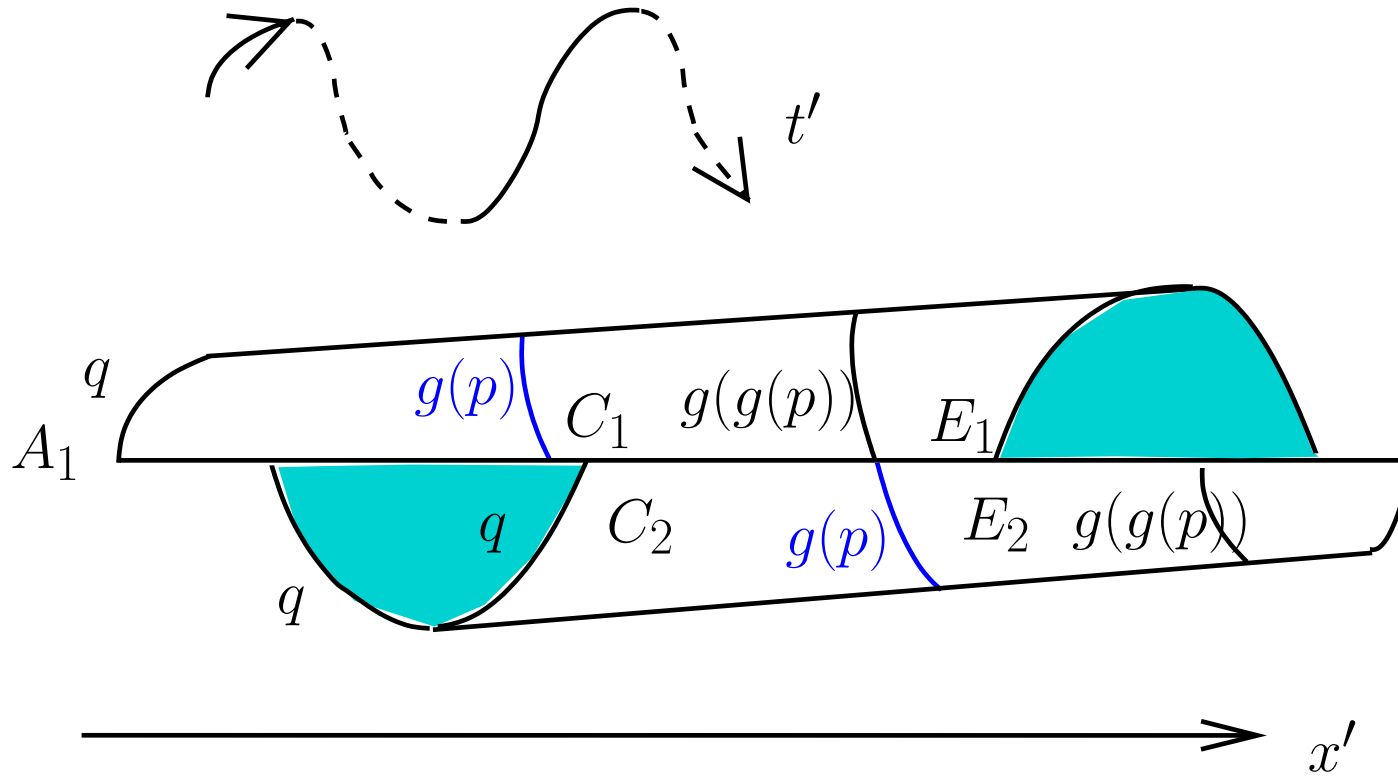




SR: twins paradox



SR: twins paradox



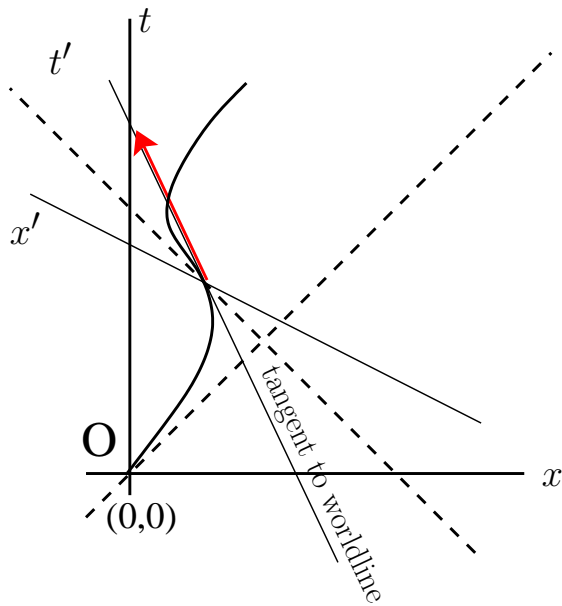
Roukema & Bajtlik 2008, MNRAS, 390, 655
[arXiv:astro-ph/0612155](https://arxiv.org/abs/astro-ph/0612155)

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with $(t, x, y, z)^T$ coord system

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with (t, x, y, z) coord system



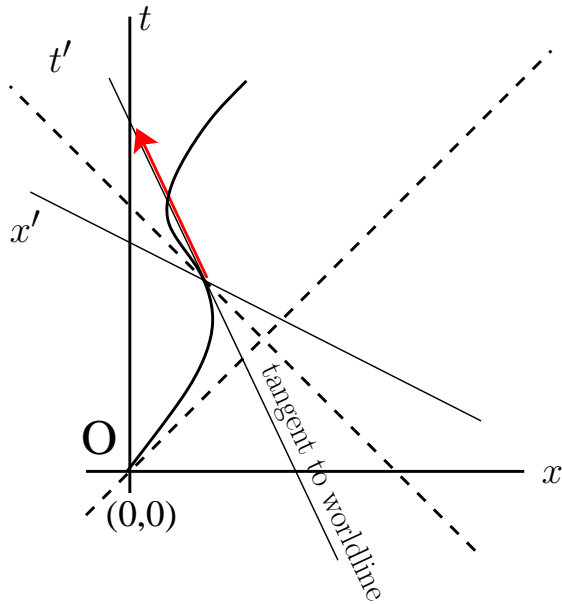
- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with (t, x, y, z) coord system

$$(u^t, u^x) := \left(\frac{d}{d\tau} t(\tau), \frac{d}{d\tau} x(\tau) \right)$$

w:four-velocity



- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector =

tangent to worldline

SR: four-velocity, four-momentum

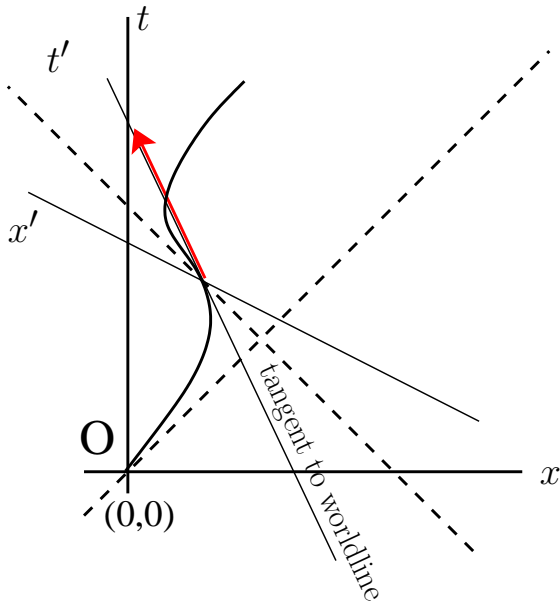
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w: four-velocity

similarly $(u^{t'}, u^{x'}) =$

$$\left(\frac{d}{d\tau} t'(\tau), \frac{d}{d\tau} x'(\tau) \right)$$

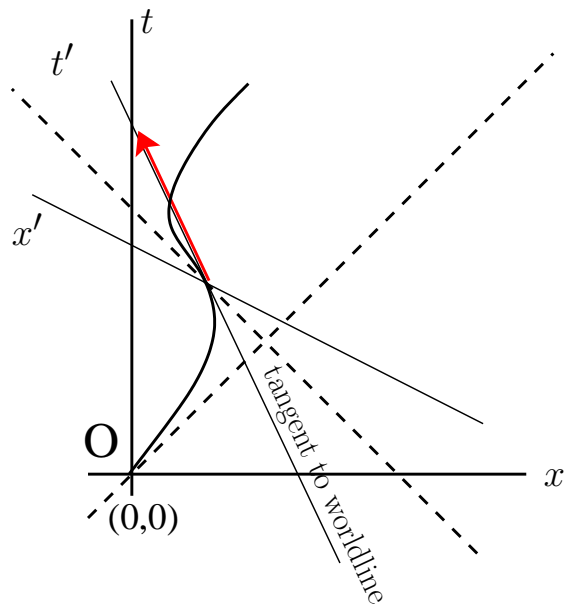


in (t, x) spacetime

2-plane, extend from

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with (t, x, y, z) coord system



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w:four-velocity

similarly $(u^{t'}, u^{x'}) =$
 $\left(\frac{d}{d\tau} t'(\tau), \frac{d}{d\tau} x'(\tau) \right) = (1, 0)$

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

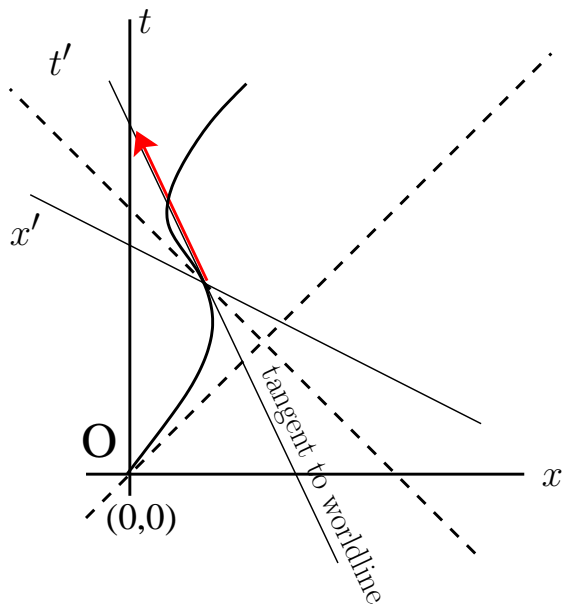
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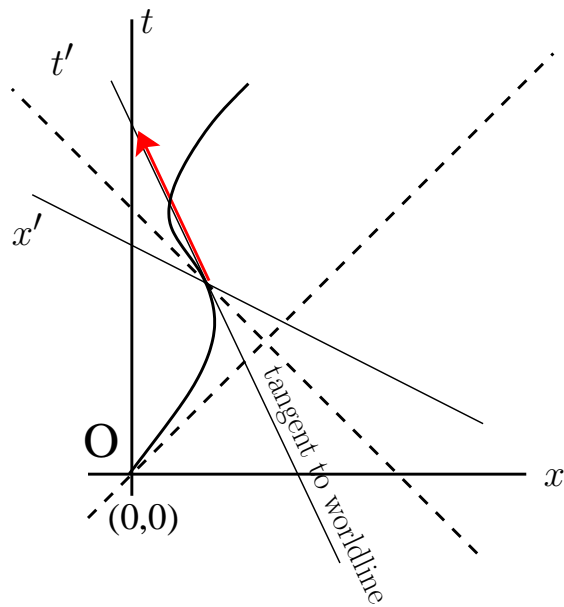
want \vec{u} Lorentz invariant $\Rightarrow (u^t, u^x)^T = \Lambda^{-1}(1, 0)^T$



- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector

SR: four-velocity, four-momentum

choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with (t, x, y, z) coord system



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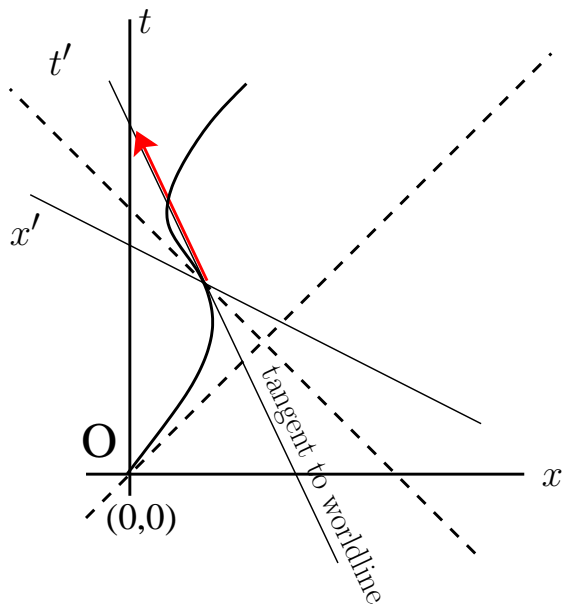
similarly $(u^{t'}, u^{x'}) = \left(\frac{d}{d\tau} t'(\tau), \frac{d}{d\tau} x'(\tau) \right) = (1, 0)$

want \vec{u} Lorentz invariant
 $\Rightarrow (u^t, u^x)^T = \Lambda^{-1}(1, 0)^T = \gamma(1, \beta)^T$

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

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choose x axis so that 3-velocity $u_{\text{Galilean}} = (\beta, 0, 0)^T$ for observer with (t, x, y, z) coord system



$$(u^t, u^x) := \left(\frac{d}{d\tau} t(\tau), \frac{d}{d\tau} x(\tau) \right)$$

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similarly $(u^{t'}, u^{x'}) = \left(\frac{d}{d\tau} t'(\tau), \frac{d}{d\tau} x'(\tau) \right) = (1, 0)$

want \vec{u} Lorentz invariant
 $\Rightarrow (u^t, u^x)^T = \Lambda^{-1}(1, 0)^T = \gamma(1, \beta)^T$

4D: $\vec{u} = \gamma(1, \beta^x, \beta^y, \beta^z)^T$

notation in this pdf:

$\vec{u} = 4\text{-vector}$, $(^3)\vec{u} = \text{spatial component}$

- in (t, x) spacetime 2-plane, extend from scalar speed β to spacetime vector = tangent to worldline

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of \vec{u} the same as the non-relativistic velocity?



SR: four-velocity, four-momentum

Is the 3-component (spatial component) of \vec{u} the same as the non-relativistic velocity?

$${}^{(3)}\vec{u} = \frac{d}{d\tau}(x, y, z)^T$$

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of \vec{u} the same as the non-relativistic velocity?

$$\begin{aligned} (3) \vec{u} &= \frac{d}{d\tau} (x, y, z)^T \\ &= \gamma \frac{d}{dt} (x, y, z)^T \end{aligned}$$

SR: four-velocity, four-momentum

Is the 3-component (spatial component) of \vec{u} the same as the non-relativistic velocity?

$$(3) \vec{u} = \frac{d}{d\tau} (x, y, z)^T$$

$$= \gamma \frac{d}{dt} (x, y, z)^T$$

$$\neq \frac{d}{dt} (x, y, z)^T \text{ except if } \beta = 0 \Leftrightarrow \gamma = 1$$

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momentum: $\vec{p} := m\vec{u} = m\gamma(1, \beta^x, \beta^y, \beta^z)^T$, where $m =$
constant w:invariant mass

x ... = tensor-style component notation, not powers

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What does the time component of momentum $= p^0 = m\gamma$
mean physically?

- first look at spatial component in a given ref. frame

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let us define 4-acceleration, 4-force



SR: four-velocity, four-momentum

$$(u^t, u^x) := \left(\frac{d}{d\tau} t(\tau), \frac{d}{d\tau} x(\tau) \right)$$



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SR: invariance of ${}^{(4)}u$, ${}^{(4)}a$, ${}^{(4)}f$



Euclidean norm: $\|\vec{x}\|^2 = \sum_{\mu} (x^{\mu})^2$



SR: invariance of ${}^{(4)}u$, ${}^{(4)}a$, ${}^{(4)}f$



Minkowski pseudo-norm: $\|\vec{x}\|^2 = \sum_{\mu,\nu} \eta_{\mu\nu} x^\mu x^\nu$



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Minkowski pseudo-norm: $\|\vec{x}\|^2 = \eta_{\mu\nu} x^\mu x^\nu$

w:Einstein summation sum is implicit



SR: invariance of $(4)u$, $(4)a$, $(4)f$



Minkowski pseudo-norm: $\|\vec{x}\|^2 = -x^0x^0 + \delta_{ij}x^ix^j$

$\delta_{ij} = 1$ if $i = j$, otherwise $= 0$; $i, j \in 1, 2, 3$

invariance: $\|\vec{x}\|^2 =$ same in all reference frames

sign convention: $(-, +, +, +)$ or $(+, -, -, -)$ are common



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similarly: $\|\vec{a}\|^2$, $\|\vec{f}\|^2$ **invariant**



SR: energy: varies with ref frame

Newtonian $K = (1/2)m\beta^2 = 0$ in rest frame



SR: energy: varies with ref frame



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4-force \vec{f} is invariant, but

3-force usually *defined* to be frame-dependent:

$$\text{3-force} := \frac{d}{dt}\vec{p} \neq \frac{d}{d\tau}\vec{p}$$



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in (x, t) frame,

$K =$ work done



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$$\begin{aligned} &= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot d\vec{x} \\ &= \int_0^{\beta_2} \frac{d}{dt} \vec{p} \cdot d\vec{x} \end{aligned}$$

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$$\begin{aligned} &= \int_0^{\beta_2} \frac{{}^{(3)}\vec{f}}{\gamma} \cdot d\vec{x} \\ &= \int_0^{\beta_2} \frac{d}{dt}(m\beta\gamma) dx \end{aligned}$$

(assume ${}^{(3)}\vec{f}/\gamma \parallel \vec{x}$)

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$$= m \int_{\gamma=1}^{\gamma=\gamma_2} [\beta^2 + (1 - \beta^2)] d\gamma$$

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$$\Rightarrow K + m = m\gamma \text{ drop "2"}$$

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so $p^0 = \text{kinetic energy} + \text{rest mass}$



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Does small β limit agree with Newtonian K ?



SR: energy: varies with ref frame



Does small β limit agree with Newtonian K ?

momentum time component:

$$p^0 = m\gamma = m(1 - \beta^2)^{-1/2}$$

$$= m[1 - (1/2)(-\beta^2) + \mathcal{O}(\beta^4)] \text{ if } \beta \ll 1$$



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Yes.



SR: $\vec{p} \dots$: invariant or not?

momentum: $\vec{p} = m\gamma(1, \beta^x, \beta^y, \beta^z)^T$

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$m =$ w:invariant mass \equiv rest mass: invariant

AND conserved (in interactions): $\|\vec{p} + \vec{q}\|^2 = \|\vec{r}\|^2$

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WARNING: assume that 4-momentum vectors at different space-time positions can be parallel-transported; not the case in curved spacetime

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vector space $\Rightarrow p^i + q^i = r^i$ ($i = 1, 2, 3$)



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= conservation of (relativistic) 3-momentum (Newtonian: conserved)

but NOT invariant (Newtonian: not invariant)

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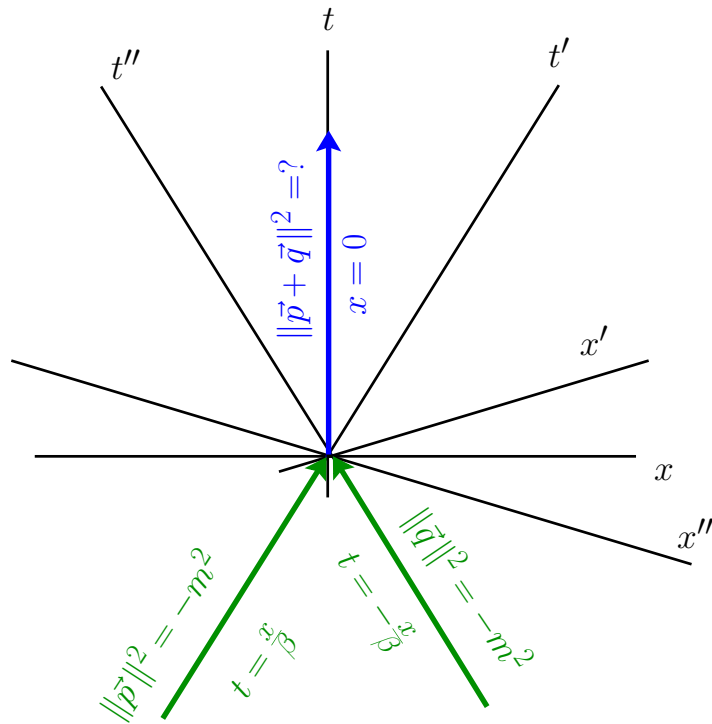
vector space $\Rightarrow p^0 + q^0 = r^0$

= conservation of (relativistic) “total energy” = $m + K$
(Newtonian: m conserved, K not conserved, $K +$
potential energy conserved)

but NOT invariant (Newtonian: m invariant, K not invariant)

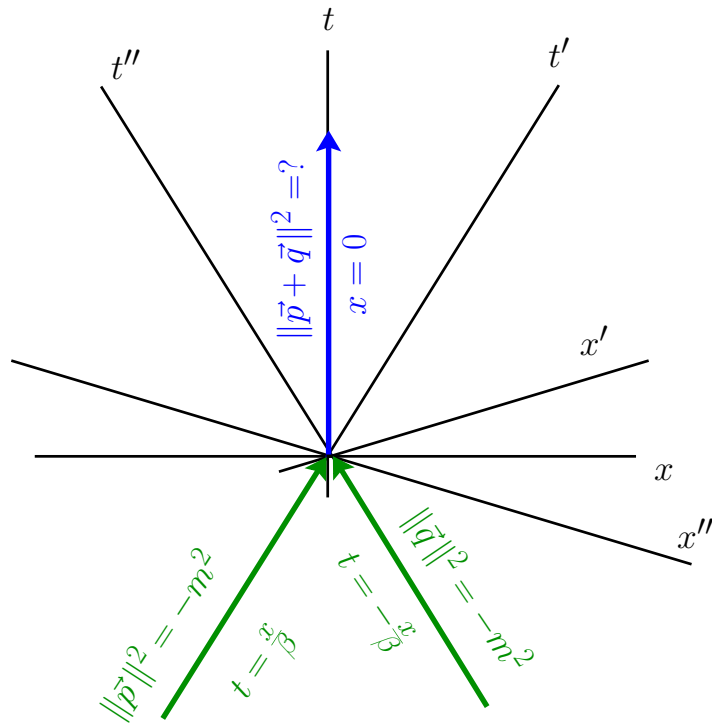


SR: $\vec{p} \dots$: invariant or not?





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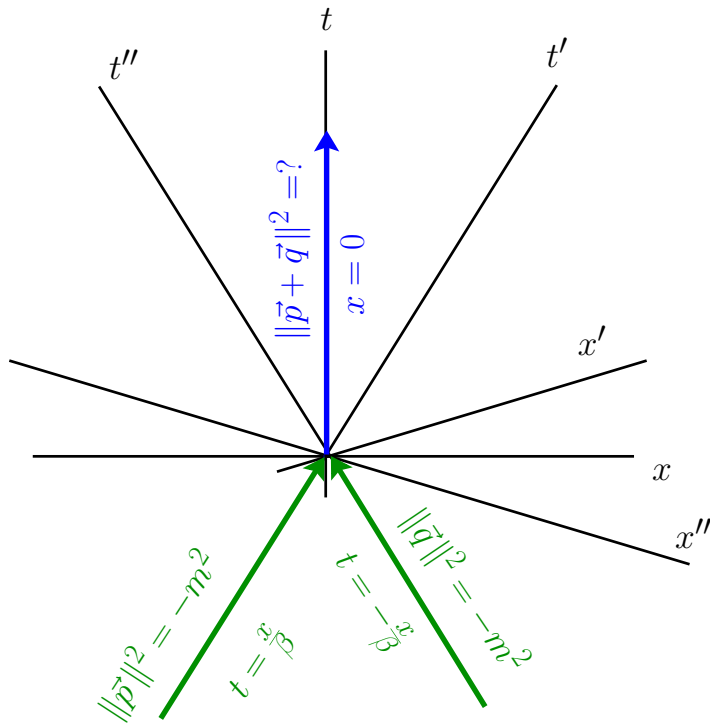


$$\vec{p} = m(\gamma, -\beta\gamma, 0, 0)^T$$





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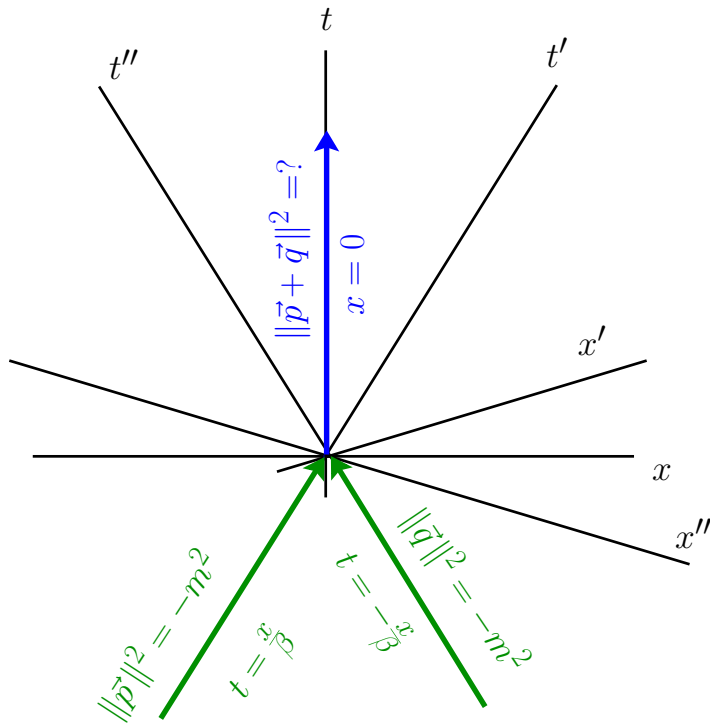


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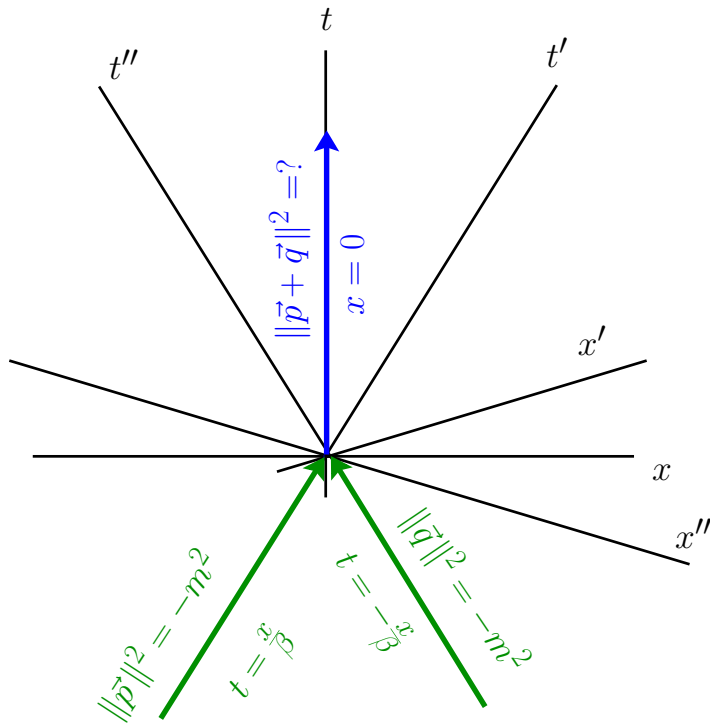


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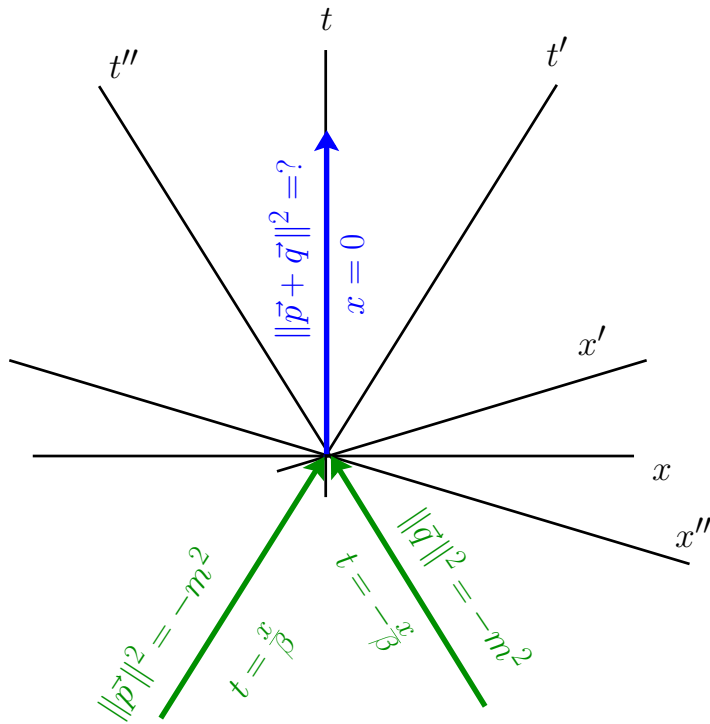
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$$\vec{r} = m[2\gamma, (-\beta + \beta)\gamma, 0, 0]^T$$

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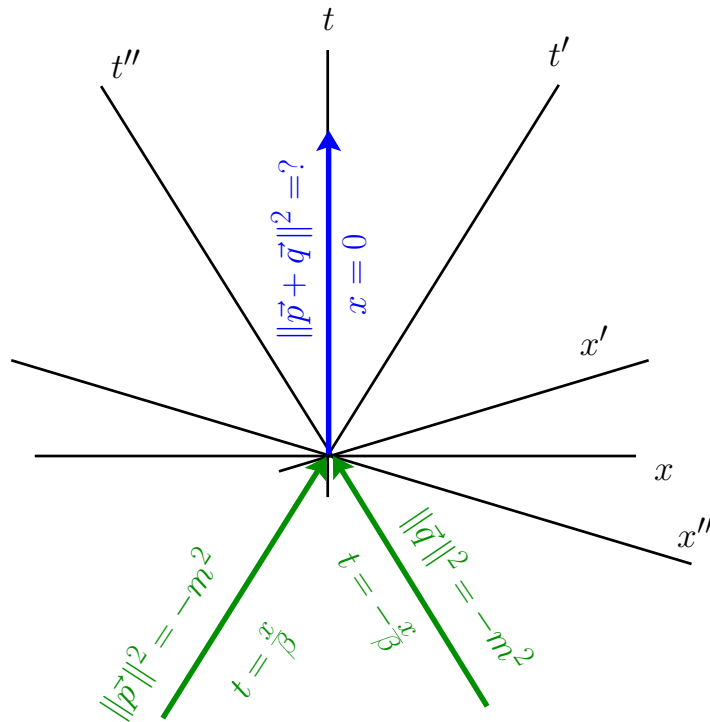


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$$\vec{r} = \vec{p} + \vec{q} = 2m\gamma(1, 0, 0, 0)^T$$

SR: $\vec{p} \dots$: invariant or not?



system rest mass before and after: $2m\gamma$

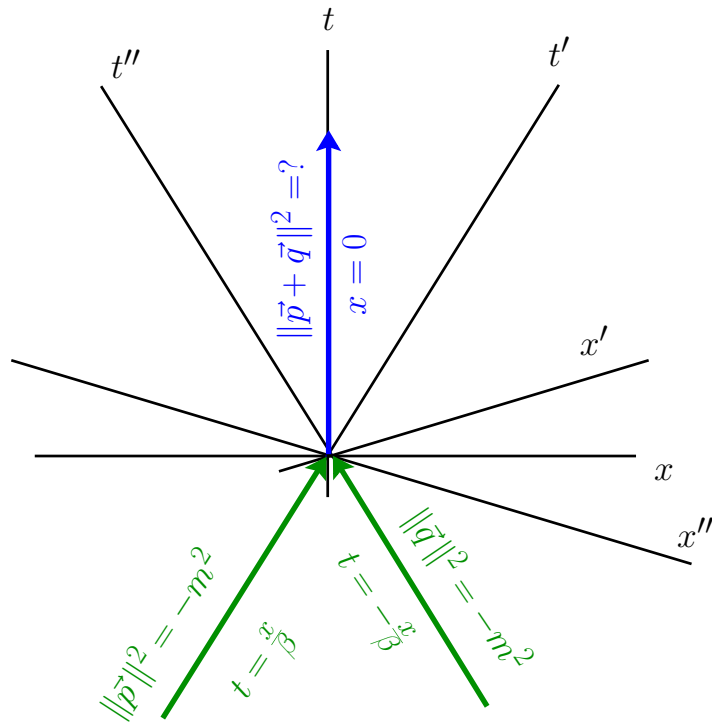
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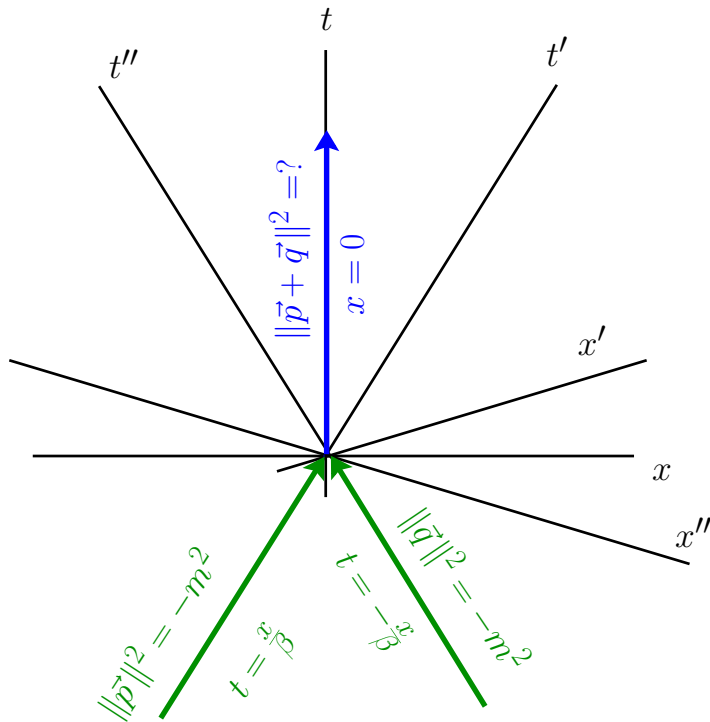
system rest mass before and after: $2m\gamma$
 rest masses in many different frames:
 $m + m \neq 2m\gamma$ if $\gamma \neq 1$

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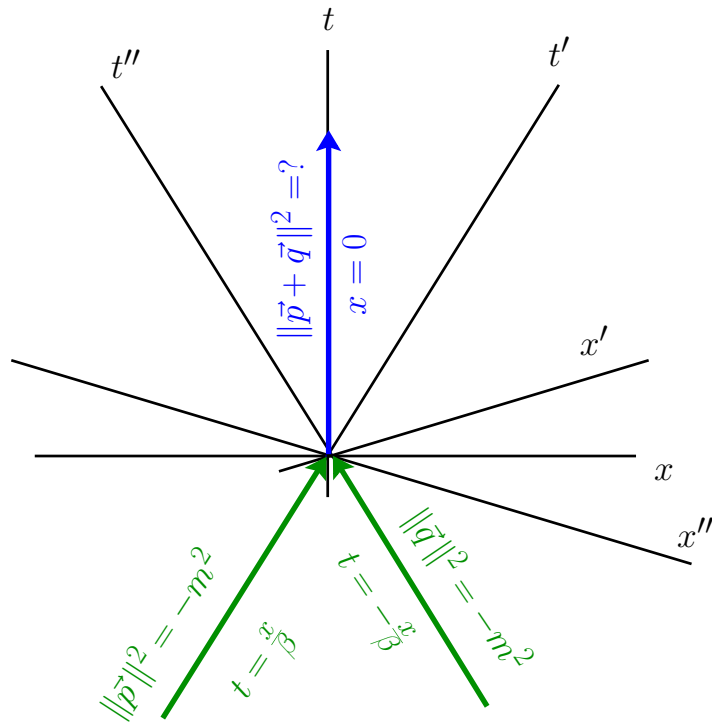
$$\sqrt{-\|\vec{p}\|^2} + \sqrt{-\|\vec{q}\|^2} \neq \sqrt{-\|\vec{r}\|^2}$$

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$$\sqrt{-\|\vec{p}\|^2} + \sqrt{-\|\vec{q}\|^2} \neq \sqrt{-\|\vec{r}\|^2}$$

system mass is invariant, but can be divided into p^0 and $p^i, i \in \{1, 2, 3\}$ components in many different ways

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+ maxima - component tensor packet ctensor





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